Employing Auctions to Allocate Scarce Inputs

by

John W. Mayo* and David E. M. Sappington**

Abstract

We examine when an unfettered auction will ensure the welfare-maximizing allocation of a scarce input in a setting where suppliers of differentiated products engage in price competition after acquiring the input at auction. A supplier values the input both because it enables the supplier to enhance the quality of its products and reduce its production cost and because, once acquired, the input is unavailable to rivals. We find that an unfettered auction often ensures the welfare-maximizing allocation of an input increment, and is particularly likely to do so when, for instance, retail customers become more concerned with vertical dimensions of product quality and less concerned with horizontal dimensions of product quality.

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* Georgetown University.

** University of Florida.

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1 Introduction

It is well known that a vertically integrated firm might seek to deny or denigrate access to an upstream input in order to foreclose downstream rivals from operating in lucrative retail markets. It is also well known that a monopolist typically is willing to pay more than a potential entrant for an essential input because, by foreclosing entry, the monopolist can secure its monopoly profit whereas an entrant can gain at most its share of a smaller duopoly profit.[1]

Recently, foreclosure concerns have expanded to the domain of auctions. For instance, some have questioned whether leading suppliers of wireless communications services might outbid smaller rivals in auctions of scarce radio spectrum primarily to limit the ability of the smaller rivals to develop into effective competitors (Department of Justice, 2013). Similarly, the Supreme Court has considered the possibility of "predatory bidding," whereby an input purchaser pays high prices in order to eliminate downstream competition from other input purchasers.[2] These foreclosure concerns naturally arise in the context of input auctions because, in additional to enabling a retail supplier to enhance the quality of its products and/or reduce its production costs, a scarce input, once acquired, is unavailable to rivals. Eso et al. (2010, p. 542) go so far as to suggest that "allocating input(s) through efficient auctions may be misguided when bidders are competing firms."

Such concerns raise important issues for both our understanding of the resource allocation properties of auctions and for public policy since auctions are commonly employed to allocate scarce inputs in practice. To illustrate, auctions have been employed to allocate billions of dollars of spectrum among suppliers of wireless communications services since the mid-1990s.[3] Timber harvesting and oil drilling rights also are typically allocated to suppliers of

[1] Corresponding considerations explain why a monopolist may engage in preemptive patenting to exclude rivals (e.g., Gilbert and Newbery, 1982). Rey and Tirole (2007) provide a comprehensive review of the literature on foreclosure.

[2] *Weyerhauser Co. v. Ross-Simmons Hardwood Lumber Co.* 127 S. Ct. 1069, 1078 (2007). Blair and Lopatka (2008) provide an informative discussion of this issue.

[3] For discussions, see McAfee and McMillan (1996), Kwerel and Rosston (2000), Hazlett and Munoz (2009), and Cramton et al. (2011), for example.

wood and oil products via auction.[4]

In light of concerns about the performance of input auctions and their prevalence in practice, it is important to better understand when auctions will and will not ensure the welfare-maximizing allocation of scarce inputs. We examine the performance of what we call unfettered input auctions, which are auctions that ensure an input is allocated to the bidder that values it most highly.[5] The bidders in our model engage in Hotelling price competition after the auction concludes. The input being auctioned can lower a firm's production cost and enhance customer valuation of its product. To illustrate, the input might be spectrum that enables a supplier of wireless communications service to increase the speed and reliability of its service and thereby reduce its customer acquisition and retention costs. Each of the firms in our model has an initial endowment of the input, and the incremental amount of the input that is being auctioned is relatively small. Consequently, no firm can totally preclude the operation of its rivals even if it were to acquire all of the available input.

In this setting, each firm derives both a "use value" and a "foreclosure value" from input increments. The use value arises because the firm that acquires the input increment can employ it to enhance its competitive position.[6] The foreclosure value arises because, by acquiring an increment of a scarce input, a firm precludes its rivals from acquiring the increment.[7] Such preclusion does not foreclose the rival in the traditional sense of driving the rival from the market, but rather in the sense of preventing the rival from employing the increment to improve its competitive position.

[4] See Hendricks et al. (1994), Haile (2001), and Athey et al. (2013), for example.

[5] For example, first-price and second-price auctions with no bidder subsidies generally have this feature. The unfettered input auctions that we analyze are isomorphic to the efficient capacity auctions that Eso et al. (2010, p. 525) consider. The authors define an efficient capacity auction to be one that allocates "each unit of capacity to the firm that values it the most."

[6] Formally, a firm's "competitive position" is the difference between the valuation that customers place on the firm's product and the firm's unit cost of production.

[7] Our nomenclature parallels that of the Department of Justice (2013, p. 10) which, in the context of spectrum auctions, observes, "... the private value [of spectrum] for incumbents ... includes not only the revenue from use of the spectrum but also any benefits gained by preventing rivals from improving their services and thereby eroding the incumbents' existing businesses. The latter might be called called 'foreclosure value' as distinct from 'use value'."

We find that although an input increment has foreclosure value in this sense, this value is not always greatest for the firm with the largest market share. In particular, although a large incumbent supplier with substantial monopoly profit to protect might outbid a potential entrant for a scarce input in order to entirely preclude the entrant's operation, a duopolist with the largest market share and profit may not outbid its rival in an auction for an increment of the scarce input. The smaller, less-profitable firm may outbid its rival if the input increment lowers cost and enhances customer valuation more rapidly for the smaller firm than for the larger firm.

We also find that an unfettered auction for the input increment will always generate the welfare-maximizing input allocation if the input increment increases the competitive position of the two rivals at the same rate. This is the case even if one firm initially enjoys a much larger market share and profit than its rival.

In addition, we find that an unfettered auction typically will produce the welfare-maximizing allocation of the input when consumers are concerned primarily with the vertical dimensions of product quality rather than the corresponding horizontal dimensions. This might be the case, for example, when consumers of wireless communications services value highly such service features as call clarity, the frequency of dropped calls, and geographic service coverage – features that are affected by the allocation of the input (spectrum) – but place relatively little value on horizontal dimensions of product quality (e.g., handset color or the proximity of a firm's showrooms to the city center) that are not affected by the input allocation.

An auction may fail to generate the welfare-maximizing allocation of an input increment when the increment would increase relatively slowly the competitive position of a firm with a moderately large market share. In this case, the larger firm may acquire the input in an unfettered auction even though welfare would be higher if the smaller firm secured the input. In principle, a bid credit for the smaller firm could be implemented to ensure the welfare-maximizing allocation of the input increment. (A bid credit reflects the fraction of its bid that a firm is not required to pay if it wins the auction for the input increment.)

However, in contrast to its typical design in practice, the appropriate bid credit does not reflect the difference in the profitabilities or the market shares of the bidding firms. Instead, it reflects the extent to which the input increment would enhance the competitive position of the smaller firm more than it would enhance the competitive position of the larger firm.

We also find that welfare can decline when a firm with a very small market share acquires more of the input, even if such acquisition does not preclude the rival firm from acquiring more of the input. (The welfare decline arises when the increased profit of the small competitor is outweighed sufficiently by the corresponding reduction in the large firm's profit.) However, unfettered auctions will avoid such welfare-reducing allocations of input increments.

Our work complements and extends the contributions of its predecessors in two primary respects. First, we extend the extensive literature on foreclosure to consider how foreclosure motives affect the ability of input auctions to generate the welfare-maximizing distribution of inputs.[8] Cramton et al. (2011, p. S168) note that auctions may fail to promote economic efficiency because "an incumbent will include in its private value not only its use value of the [scarce input] but also the value of keeping [it] from a competitor." Thus, the literature has recognized this foreclosure value. However, studies in the literature typically do not develop the implications of this value in a comprehensive model that includes the post-auction interaction among retail competitors after they participate in the auction.

Because our analysis incorporates such a model, we are able to determine explicitly how an input acquisition benefits a competitor by improving its own competitive position and by foreclosing its rivals from the opportunity to enhance their competitive positions. Consequently, we are able to identify the characteristics of firms and industry conditions that enhance or limit the propensity of unfettered auctions to generate the welfare-maximizing allocation of inputs.

Second, most studies of input auctions that do model the post-auction interaction among bidders consider Cournot competition among suppliers of a homogeneous retail product. In

[8] It is noteworthy in this regard that Rey and Tirole's (2007) comprehensive review of the foreclosure literature cites no studies that examine how retail suppliers might take advantage of input auctions to foreclose rivals.

this context, Borenstein (1988), for instance, demonstrates that the profit a firm secures from a license to operate upstream can differ systematically from the total surplus it generates downstream. In such cases, auctions of licenses often fail to generate welfare-maximizing outcomes. In a setting more similar to ours, McAfee (1998) highlights the important role that firms' initial capacities can play in determining which firm wins the auction for incremental capacity.[9] McAfee finds that small, capacity-constrained firms often will outbid larger, unconstrained firms in part because each unconstrained bidder cannot capture the full increase in industry profit that arises when small producers are precluded from acquiring additional capacity.[10] Burguet and McAfee (2009) report that auctions of operating licenses maximize consumers' surplus even when suppliers face binding financing constraints, provided consumer demand for the homogeneous retail product is sufficiently elastic. Eso et al. (2010) analyze auctions that allocate capacity increments to the firms that value them most highly. The authors show that even when all suppliers are symmetric *ex ante*, capacity increments may be allocated asymmetrically, so the equilibrium downstream industry configuration entails one large firm with no capacity constraint facing smaller, capacity-constrained rivals.[11]

Our analysis of price competition among suppliers of differentiated products admits insights that models of Cournot competition cannot provide.[12] In particular, we explain how firm bidding behavior and input allocations are influenced by consumer preferences for the distinct products sold by different retail suppliers. We also explain how input allocations are

[9] Hendricks and McAfee (2010) extend models with Cournot competition among users of an input to include competition among suppliers of the homogeneous input. The authors allow firms to be both buyers and sellers of the input. Equilibrium allocations are determined by (strategic, endogenous) supply and demand in their model, rather than by auctions.

[10] McAfee (1998) also analyzes a model in which firms engage in Cournot competition after bidding to acquire an input that reduces a firm's total and marginal cost of production. He finds that the firm with the smallest initial endowment of the input will win the auction for the input increment.

[11] Eso et al. (2010) also analyze a model of price competition between capacity-constrained suppliers of differentiated products. Their focus in this analysis, too, is on potential asymmetries in the post-auction size distributions of industry suppliers.

[12] Burguet and McAfee (2009, n. 8) observe that "differentiated product models are notoriously challenging to analyze. However, the analysis of such models represents the natural next step."

affected by the relative importance of vertical and horizontal dimensions of product quality.

Like several of its predecessors (e.g., Katz and Shapiro, 1986; Jehiel et al., 1996; Jeheil and Moldovanu, 1996; Das Varma, 2002) our analysis entails auction externalities in the sense that the payoffs of bidders who do not win the auction are directly affected by the auction outcome.[13] Although these studies recognize that input auctions may fail to allocate resources efficiently,[14] the principal focus of many of these studies is on characterizing the auction format that maximizes the seller's payoff. Jehiel and Moldovanu (2000) provide a related investigation that focuses on the design of reserve prices and entry fees in second-price auctions. Although our primary focus is not on auction design, we do consider how bidding credits can be designed to ensure the welfare-maximizing allocation of inputs when private and social valuations of inputs diverge. Our finding that substantial information is required to ensure this outcome is consistent with the literature's message regarding the complexity of auction design in the presence of externalities.

Our analysis proceeds as follows. Section 2 describes the key elements of our model of retail competition and characterizes equilibrium outcomes, taking as given the prevailing allocation of the scarce input.[15] Section 3 examines the private and the social gains that arise as a competitor secures additional units of the input. Section 4 identifies the conditions under which an unfettered auction will, and will not, ensure a welfare-maximizing allocation of an input increment. Section 5 summarizes our main conclusions and discusses extensions of our analysis.

[13] Jehiel and Moldovanu (2000, p. 768) consider externalities to be present when "even agents who are not directly involved in a transaction are indirectly affected by its outcome."

[14] Jeheil and Moldovanu (2003, p. 271) observe that "when the assets for sale ... are inputs that will subsequently be used by the successful bidders in imperfect competition with each other ... auctions can behave in surprisingly problematic ways."

[15] The more tedious calculations that underlie (and extend) the findings in section 2 are presented in the Appendix.

2 Equilibrium Outcomes in the Model

We consider a setting where two firms engage in price competition after acquiring a key input (e.g., spectrum) at auction. The input that a firm acquires reduces its production costs and/or enhances consumer valuations of its product. We will denote by v_i the value that each consumer derives from purchasing one unit of firm $i \in \{1, 2\}$'s product.[16] This product might be a subscription to the firm's wireless communications service and the functionality admitted by this subscription, for example. Customer value of firm i's product is an increasing function of the amount of the input (k_i) that firm i employs (i.e., $v_i'(k_i) > 0$). The increased value might arise, for example, because additional spectrum enables a firm to increase the speed and reliability of its wireless communications service. Additional units of the input also can reduce a firm's unit cost of serving customers (i.e., $c_i'(k_i) \leq 0$, where $c_i(k_i)$ is firm i's unit cost of production when it employs k_i units of the input).[17]

All consumers value symmetrically the product enhancement that higher levels of the input provide (e.g., faster download speeds and/or fewer dropped calls). However, consumers differ in their valuations of other elements of the firms' products. These elements include, for example, the color and design of telephone hand*sets* or the geographic locations of a firm's showrooms and service centers. To capture these different valuations, we employ the standard Hotelling model of competition and assume that potential consumers are distributed uniformly on the $[0, 1]$ interval and must travel either to point 0 to purchase the product from firm 1 or to point 1 to purchase the product from firm 2. Each consumer experiences unit transportation cost t as she travels to purchase the product. Therefore, a consumer who travels distance d to purchase the product from firm $i \in \{1, 2\}$ and pays price p_i for the good receives net utility $v_i(\cdot) - p_i - t\,d$.

To focus on the case of primary interest in which both firms serve some consumers in

[16] For simplicity, we assume that consumers place such limited value on multiple units of the product that they never purchase more than one unit.

[17] The cost saving may arise, for instance, from reduced use of less efficient inputs that a firm is compelled to employ when it has limited access to the more efficient input in question. Increased customer valuation of a firm's product also can reduce the firm's customer acquisition and retention costs.

equilibrium, we assume that the firms' costs and consumer valuations of the firms' products are not too disparate.[18] Formally, we adopt Assumption 1, which bounds the difference between the value margins of the two firms. The value margin of firm i is the difference between consumer valuation of its product and its unit cost of serving customers, i.e., $m_i(k_i) = v_i(k_i) - c_i(k_i)$.

Assumption 1. $-3t \leq m_1(k_1) - m_2(k_2) \leq 3t$ for all relevant k_1 and k_2.

To characterize equilibrium outcomes in this model, we first specify the prices that the firms will set for their products, given the levels of the input they have acquired. These prices permit a specification of the equilibrium market shares of the firms, their profits, consumers' surplus, and welfare (i.e., the sum of consumers' surplus and industry profit). After characterizing these equilibrium outcomes for a given allocation of the input, we examine the allocation that an auction will produce, and compare this allocation to the welfare-maximizing allocation.

We begin by identifying the location ($\widehat{x} \in [0, 1]$) of the consumer who is indifferent between purchasing the product from the two firms when consumers place value v_i on firm $i \in \{1, 2\}$'s product and when firm i charges price p_i for its product. This location is determined by:

$$v_1 - p_1 - t\,\widehat{x} = v_2 - p_2 - t\,[1 - \widehat{x}] \quad \Rightarrow \quad \widehat{x} = \frac{1}{2t}\,[t + v_1 - p_1 - (v_2 - p_2)]\,. \tag{1}$$

Equation (1) implies that at an interior solution with full-market coverage (in which all consumers buy a unit of the product and each firm serves some customers), firm 1 will serve all customers in the $[0, \widehat{x}]$ interval and firm 2 will serve all customers in the $(\widehat{x}, 1]$ interval.[19] Therefore, normalizing to 1 the total number of potential customers, equation (1) implies that firm 1's profit is:

[18] We also assume that $v_1(0)$ and $v_2(0)$ are sufficiently large relative to t that all consumers purchase one unit of the product in equilibrium.

[19] We assume that a customer will purchase the product from firm 1 when she is indifferent between purchasing the product from the two firms.

$$\pi_1 \;=\; [\,p_1 - c_1\,]\,\widehat{x} \;=\; \frac{1}{2\,t}\,[\,p_1 - c_1\,]\,[\,t + v_1 - p_1 - (v_2 - p_2)\,]\,. \tag{2}$$

Equation (2) implies that firm 1's profit-maximizing price is determined by:

$$\frac{\partial \pi_1}{\partial p_1} \;=\; 0 \quad\Leftrightarrow\quad p_1 \;=\; \frac{1}{2}\,[\,t + c_1 + v_1 - v_2 + p_2\,]\,. \tag{3}$$

Corresponding calculations reveal that firm 2's profit-maximizing price is determined by:

$$p_2 \;=\; \frac{1}{2}\,[\,t + c_2 + v_2 - v_1 + p_1\,]\,. \tag{4}$$

Solving equations (3) and (4) simultaneously provides explicit expressions for the prices that the firms will charge in equilibrium and the firms' corresponding price-cost margins:[20]

$$p_1^* \;=\; \frac{1}{3}\,[\,3\,t + v_1 - v_2 + 2\,c_1 + c_2\,] \quad\Rightarrow\quad p_1^* - c_1 \;=\; \frac{1}{3}\,[\,3\,t + m_1 - m_2\,] \quad \text{and} \tag{5}$$

$$p_2^* \;=\; \frac{1}{3}\,[\,3\,t + v_2 - v_1 + 2\,c_2 + c_1\,] \quad\Rightarrow\quad p_2^* - c_2 \;=\; \frac{1}{3}\,[\,3\,t + m_2 - m_1\,]\,. \tag{6}$$

Equations (5) and (6) are readily employed to show that the equilibrium outputs of firms 1 and 2 are, respectively:

$$x_1^* \;=\; \frac{1}{6\,t}\,[\,3\,t + m_1 - m_2\,] \quad \text{and} \quad x_2^* \;=\; \frac{1}{6\,t}\,[\,3\,t + m_2 - m_1\,]\,. \tag{7}$$

From equations (5) and (7), firm 1's equilibrium profit (given v_1, v_2, c_1, and c_2) is:

$$\pi_1^* \;=\; [\,p_1^* - c_1\,]\;x_1^* \;=\; \frac{1}{18\,t}\,[\,3\,t + m_1 - m_2\,]^2\,. \tag{8}$$

Corresponding calculations using equations (6) and (7) reveal that firm 2's equilibrium profit is:

$$\pi_2^* \;=\; \frac{1}{18\,t}\,[\,3\,t + m_2 - m_1\,]^2\,. \tag{9}$$

Let $\Delta \equiv m_1 - m_2$ denote the difference between the value margins of firms 1 and 2. Also let $\pi^* \equiv \pi_1^* + \pi_2^*$ denote equilibrium industry profit. Then equations (8) and (9) provide:

$$\pi^* \;=\; \frac{1}{18\,t}\,[\,3\,t + \Delta\,]^2 + \frac{1}{18\,t}\,[\,3\,t - \Delta\,]^2 \;=\; t + \frac{\Delta^2}{9\,t}\,. \tag{10}$$

Equations (5), (6), and (7) imply that equilibrium consumers' surplus is:

[20] The details of this calculation and others are provided in the Appendix. The superscript * on a variable denotes its equilibrium value.

$$CS^* = \int_0^{x_1^*} [\, v_1 - p_1^* - t\, x \,]\, dx + \int_{x_1^*}^{1} [\, v_2 - p_2^* - t\,(1-x) \,]\, dx$$

$$= \frac{m_1}{6\,t} [\, 3\,t + \Delta \,] + \frac{m_2}{6\,t} [\, 3\,t - \Delta \,] - \frac{5\,t}{4} - \frac{5\,\Delta^2}{36\,t}. \qquad (11)$$

Equations (10) and (11) imply that equilibrium welfare is:

$$W^* = CS^* + \pi^* = \frac{m_1}{6\,t} [\, 3\,t + \Delta \,] + \frac{m_2}{6\,t} [\, 3\,t - \Delta \,] - \frac{t}{4} - \frac{\Delta^2}{36\,t}. \qquad (12)$$

3 Input Allocations

We now characterize the welfare-maximizing allocation of an input increment and the allocation that an auction will produce. To begin, observe that if an increment of the input is awarded to, say, firm 1, then the increment is not awarded to firm 2. Therefore, when firm 1 outbids firm 2 for an increment of the input at auction, firm 1 acquires the input increment itself and simultaneously precludes firm 2 from acquiring the increment. Consequently, the rate at which firm 1's equilibrium profit (π_1^*) increases as it acquires more of the input at auction is the sum of: (i) $\frac{\partial \pi_1^*}{\partial k_1}$, the rate at which π_1^* increases as k_1 increases; and (ii) $-\frac{\partial \pi_1^*}{\partial k_2}$, the rate at which π_1^* increases as k_2 decreases.[21]

$\frac{\partial \pi_1^*}{\partial k_1}$ can be viewed as the marginal "use value" that firm 1 derives from the input. This marginal use value reflects the rate at which additional units of the input enable firm 1 to increase its profit by enhancing its competitive position (i.e., by reducing its unit cost of production and/or enhancing customer valuation of its product). $-\frac{\partial \pi_1^*}{\partial k_2}$ can be viewed as the marginal "foreclosure value" that firm 1 derives from the input. This foreclosure value reflects the increase in firm 1's profit that arises because because firm 2 does not acquire more of the input, and therefore does not have the corresponding opportunity to enhance its competitive position.

Equation (8) admits explicit expressions for the marginal use value and foreclosure value that firm 1 derives from the input. Formally:

[21] McAfee (1998) provides a corresponding discussion.

$$\frac{\partial \pi_1^*}{\partial k_1} = \frac{1}{9\,t}\,[\,3\,t + m_1 - m_2\,]\,\frac{\partial m_1}{\partial k_1} \quad \text{and} \quad -\frac{\partial \pi_1^*}{\partial k_2} = \frac{1}{9\,t}\,[\,3\,t + m_1 - m_2\,]\,\frac{\partial m_2}{\partial k_2}\,. \tag{13}$$

Expression (13) implies that the marginal use value that firm 1 derives from the input increases as $\frac{\partial m_1}{\partial k_1}$ and $m_1 - m_2$ increase, *ceteris paribus*. When $\frac{\partial m_1}{\partial k_1}$ is large, additional units of the input enable firm 1 to reduce its unit production cost and/or increase customer valuation of its product substantially. When $m_1 - m_2$ is large, firm 1 has achieved a relatively high value margin and so enjoys a relatively large market share (recall equation (7)). Consequently, additional units of the input enable firm 1 to profit by reducing its unit cost of serving its many customers and/or by increasing the attraction of its product to its large customer base.

Expression (13) *also* implies that the marginal foreclosure value firm 1 derives from the input increases as $\frac{\partial m_2}{\partial k_2}$ and $m_1 - m_2$ increase, *ceteris paribus*. When $\frac{\partial m_2}{\partial k_2}$ is large, additional units of the input would enable firm 2 to substantially enhance its competitive position, and thereby erode firm 1's profit. This profit erosion is particularly pronounced when firm 1 has achieved a relatively high value margin and so serves a relatively large number of customers in equilibrium and also enjoys a relatively high profit margin (recall equations (5) and (7)).

The maximum amount firm 1 will pay to secure a small increment of the input rather than let firm 2 secure the increment is the sum of the marginal use value and foreclosure value that firm 1 derives from the input. From expression (13), this sum is:

$$B_1 \equiv \frac{\partial \pi_1^*}{\partial k_1} - \frac{\partial \pi_1^*}{\partial k_2} = \frac{1}{9\,t}\,[\,3\,t + m_1 - m_2\,]\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]. \tag{14}$$

Corresponding calculations reveal that the maximum amount firm 2 will pay to secure a small increment of the input rather than let firm 1 secure the increment is:

$$B_2 \equiv \frac{\partial \pi_2^*}{\partial k_2} - \frac{\partial \pi_2^*}{\partial k_1} = \frac{1}{9\,t}\,[\,3\,t + m_2 - m_1\,]\left[\frac{\partial m_2}{\partial k_2} + \frac{\partial m_1}{\partial k_1}\right]. \tag{15}$$

The firm that will win an unfettered auction for an input increment is the firm that anticipates the largest increase in its equilibrium profit from securing the increment rather than allowing its rival to do so. Therefore, equations (14) and (15) imply that, in the presence of complete information, firm 1 will win an auction for the input increment if $B_1 > B_2$.[22] This observation provides the following conclusion.

Lemma 1. *The firm with the highest value margin will win an unfettered auction for the input increment.*

Proof. Equations (14) and (15) imply that $B_1 \geq B_2$ if and only if:

$$\frac{1}{9\,t}\left[\,3\,t + m_1 - m_2\,\right]\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right] \;\geq\; \frac{1}{9\,t}\left[\,3\,t + m_2 - m_1\,\right]\left[\frac{\partial m_2}{\partial k_2} + \frac{\partial m_1}{\partial k_1}\right]$$

$$\Leftrightarrow \quad m_1 - m_2 \;\geq\; m_2 - m_1 \quad \Leftrightarrow \quad \Delta \;=\; m_1 - m_2 \;\geq\; 0\,. \quad \blacksquare \tag{16}$$

To characterize the input allocations that maximize welfare, recall that an input increment that is allocated to firm 1 cannot be simultaneously allocated to firm 2. Therefore, the rate at which equilibrium welfare (W^*) increases as more of the scarce input is allocated to firm 1 is: (i) $\frac{\partial W^*}{\partial k_1}$, the rate at which W^* increases as k_1 increases; and (ii) $-\frac{\partial W^*}{\partial k_2}$, the rate at which W^* increases as k_2 decreases. Observe from equation (12) that:

$$\frac{\partial W^*}{\partial k_1} \;=\; \frac{1}{18\,t}\,\frac{\partial m_1}{\partial k_1}\left[\,9\,t + 3\,(\,m_1 - m_2) + 2\,\Delta\,\right] \;=\; \frac{\partial m_1}{\partial k_1}\left[\frac{1}{2} + \frac{5\,\Delta}{18\,t}\right], \text{ and} \tag{17}$$

$$\frac{\partial W^*}{\partial k_2} \;=\; \frac{1}{18\,t}\,\frac{\partial m_2}{\partial k_2}\left[\,3\,(\,m_2 - m_1\,) + 3\,(\,3\,t - \Delta\,) + \Delta\,\right] = \frac{\partial m_2}{\partial k_2}\left[\frac{1}{2} - \frac{5\,\Delta}{18\,t}\right]. \tag{18}$$

Equations (17) and (18) imply that the rate at which equilibrium welfare increases as the input is allocated to firm 1 rather than firm 2 is:

$$G_1 \;\equiv\; \frac{\partial W^*}{\partial k_1} - \frac{\partial W^*}{\partial k_2} \;=\; \frac{1}{2}\left[\frac{\partial m_1}{\partial k_1} - \frac{\partial m_2}{\partial k_2}\right] + \frac{5\,\Delta}{18\,t}\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]. \tag{19}$$

Similarly, the rate at which equilibrium welfare increases as the input is allocated to firm 2 rather than firm 1 is:

[22] Here and throughout the ensuing analysis, we focus on the rate at which the input increases key variables (e.g., profit and welfare) rather than the amount by which the input increment increases these variables. This focus streamlines the formal analysis without affecting the qualitative conclusions that would arise from an analysis of small, discrete input changes.

$$G_2 \;\equiv\; \frac{\partial W^*}{\partial k_2} - \frac{\partial W^*}{\partial k_1} \;=\; -\,G_1. \tag{20}$$

Equations (19) and (20) imply that equilibrium welfare increases relatively rapidly when the input accrues to the firm that: (i) experiences a relatively rapid increase in its value margin as it secures more of the input; and (ii) has achieved a relatively high value margin and corresponding market share. In this latter sense, welfare tends to increase as the input is allocated to enhance the success of the most successful firm. This finding reflects in part the fact that industry profit increases as the input is allocated to the firm with the highest value margin. Formally, from equations (14) and (15):

$$\frac{\partial \pi^*}{\partial k_1} - \frac{\partial \pi^*}{\partial k_2} \;=\; B_1 - B_2 \;=\; \frac{2}{9\,t}\left[\, m_1 - m_2 \,\right] \left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2} \right] \gtreqless 0 \;\Leftrightarrow\; m_1 - m_2 \gtreqless 0\,. \tag{21}$$

Equations (19) and (20) imply that welfare increases more rapidly when firm 1, rather than firm 2, secures the input increment if $G_1 > G_2$. This observation provides the following conclusion.

Lemma 2. *Welfare is highest when the input increment is allocated: (i) to firm 1 if* $\Delta = m_1 - m_2 \geq \frac{9\,t}{5}\,D$*; and (ii) to firm 2 if* $\Delta \leq \frac{9\,t}{5}\,D$*, where* $D \equiv \frac{\frac{\partial m_2}{\partial k_2} - \frac{\partial m_1}{\partial k_1}}{\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}}$.

Proof. From equations (19) and (20):

$$G_1 \;\geq\; G_2 \;\;\Leftrightarrow\;\; G_1 \;\geq\; -\,G_1 \;\;\Leftrightarrow\;\; G_1 \;\geq\; 0$$

$$\Leftrightarrow\;\; \frac{5\,\Delta}{9\,t} \left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2} \right] \;\geq\; \frac{\partial m_2}{\partial k_2} - \frac{\partial m_1}{\partial k_1} \;\;\Leftrightarrow\;\; \Delta \;=\; m_1 - m_2 \;\geq\; \frac{9\,t}{5}\,D\,. \;\;\blacksquare \tag{22}$$

Lemma 2 reports that when an input increment increases, say, firm 2's value margin more rapidly than it increases firm 1's value margin (i.e., when $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$ and so $D > 0$), welfare typically is highest when the increment is awarded to firm 2. The only exception is when firm 1's value margin substantially exceeds firm 2's value margin (i.e., when $m_1 - m_2 > \frac{9\,t}{5}\,D$). When m_1 substantially exceeds m_2, firm 1 serves many more customers than firm 2 in equilibrium. (Recall equation (7).) Consequently, welfare is highest when firm 1 employs the input increment to enhance the value that its substantial customer base derives from its product and/or to reduce the unit cost of serving this large customer base, even though the increment would increase firm 2's value margin more than it increases firm 1's value margin.

4 Main Findings

Before determining when an unfettered auction will secure the welfare-maximizing allocation of an input increment, we examine the effects of an increase in the amount of the input that one firm secures with no corresponding reduction in the amount of the input available to the other firm.

Proposition 1. *An increase in k_1: (i) increases π_1^* and reduces π_2^*; (ii) increases π^* if and only if $m_1 > m_2$; (iii) reduces p_2^*; (iv) reduces p_1^* if and only if $\left|\frac{\partial c_1}{\partial k_1}\right| > \frac{1}{2}\frac{\partial v_1}{\partial k_1}$; (v) increases $p_1^* - p_2^*$ if and only if $\frac{\partial v_1}{\partial k_1} > \frac{1}{2}\left|\frac{\partial c_1}{\partial k_1}\right|$; (vi) increases CS^*; and (vii) increases W^* if and only if $m_1 - m_2 > -\frac{9}{5}t$.*

<u>Proof</u>. Conclusion (i) reflects expression (13) and Assumption 1. Conclusion (ii) follows from equation (10).

From equation (5), $\frac{\partial p_1^*}{\partial k_1} = \frac{1}{3}\left[\frac{\partial v_1}{\partial k_1} + 2\frac{\partial c_1}{\partial k_1}\right] \gtreqless 0 \;\Leftrightarrow\; \frac{\partial v_1}{\partial k_1} \gtreqless 2\left|\frac{\partial c_1}{\partial k_1}\right|$. From equation (6), $\frac{\partial p_2^*}{\partial k_1} = -\frac{1}{3}\left[\frac{\partial v_1}{\partial k_1} - \frac{\partial c_1}{\partial k_1}\right] < 0$. Therefore, $\frac{\partial\left(p_1^* - p_2^*\right)}{\partial k_1} = \frac{1}{3}\left[2\frac{\partial v_1}{\partial k_1} + \frac{\partial c_1}{\partial k_1}\right] \gtreqless 0 \;\Leftrightarrow\; \frac{\partial v_1}{\partial k_1} \gtreqless \frac{1}{2}\left|\frac{\partial c_1}{\partial k_1}\right|$.

From equation (11) and Assumption 1, $\frac{\partial CS^*}{\partial k_1} = \frac{1}{6\,t}\frac{\partial m_1}{\partial k_1}\left[m_1 - m_2 + 3\,t + \Delta - \frac{5\,\Delta}{3}\right] = \frac{1}{18\,t}\frac{\partial m_1}{\partial k_1}\left[9\,t + \Delta\right] > 0$. From equation (17) and Assumption 1, $\frac{\partial W^*}{\partial k_1} = \frac{1}{18\,t}\frac{\partial m_1}{\partial k_1}\left[9\,t + 5\,\Delta\right] \gtreqless 0 \;\Leftrightarrow\; \Delta \gtreqless -\frac{9}{5}\,t\,.$ ■

For emphasis, we restate conclusion (vii) in Proposition 1 as:

Corollary 1. *Aggregate welfare declines when a firm with a sufficiently small value margin secures more of the input.*

Proposition 1 indicates that when firm 1 acquires more of the input, it will raise its price if the input increment primarily increases the value (v_1) that consumers place on its product. In contrast, firm 1 will reduce its price if the predominant effect of an increase in k_1 is to reduce c_1, its unit cost of production. Firm 2 will always reduce its price when firm

1 acquires more of the input. Firm 2 does so in order to counteract the firm 1's enhanced competitive position. These price changes and the changes in v_1 and c_1 ensure that firm 1's profit increases and firm 2's profit declines.

It is apparent that consumers' surplus will increase when firm 1 and firm 2 both reduce their prices. Consumers' surplus will also increase when firm 1 increases its price and firm 2 reduces its price. This is the case because when firm 1 increases its price, it does so by less than the increase in the value that consumers derive from its product. (Observe from expression (5) that $\frac{\partial p_1^*}{\partial k_1} \leq \frac{1}{3} \frac{\partial v_1}{\partial k_1}$.)

Corollary 1 reflects in part the fact that when the firm that acquires the input (e.g., firm 1) serves a sufficiently small share of consumers (e.g., when $m_1 < m_2 - \frac{9}{5} t$), the increase in the firm's profit is small relative to the reduction in the rival's profit.[23]

Consequently, aggregate welfare can decline when additional units of the input are awarded to a firm that serves a sufficiently small share of the market, even in the absence of any accompanying reduction in the rival firm's input.[24]

We now determine when an unfettered auction will generate the welfare-maximizing allocation of an input increment in the setting under consideration.

Proposition 2. *An unfettered auction will secure the welfare-maximizing allocation of an input increment if the increment: (i) increases the value margins of the two firms at the same rate (i.e.,* $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2}$*); or (ii) increases most rapidly the highest value margin or the value margin that is the smallest by at least* $\frac{9\,t}{5} \mid D \mid$ *(i.e., if* $\frac{\partial m_i}{\partial k_i} > \frac{\partial m_j}{\partial k_j}$ *and either* $m_i > m_j$ *or* $m_i < m_j - \frac{9\,t}{5} \mid D \mid$ *for some* $i \in \{1, 2\}$*, where* $j \neq i$*, and* $j \in \{1, 2\}$*).*

Proof. If $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2}$, then equations (19) and (20) imply that welfare increases most rapidly when the input increment is allocated to the firm with the highest value margin, which an

[23] Recall from equation (21) that firm 1's profit increases by more than firm 2's profit declines if and only if firm 1 is initially serving more customers than firm 2 (i.e., if and only if $m_1 > m_2$).

[24] Observe from equations (19) and (20), though, that aggregate welfare will not decline when the input increment is allocated so as to maximize welfare. Since $G_1 = -G_2$, if welfare declines when the increment is allocated to one firm, then welfare will increase when the input is allocated to the other firm.

unfettered auction ensures. (Recall Lemma 1.)

From Lemmas 1 and 2: (i) firm 1 will win the auction for the input increment if $\Delta > 0$; and (ii) welfare is highest when firm 1 wins the auction if $\Delta > \frac{9t}{5} D$. Therefore, the auction ensures the welfare-maximizing allocation of the input increment if $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$ and either $m_1 > m_2$ (so firm 1 wins the auction) or $m_1 < m_2 - \frac{9t}{5} \mid D \mid$ (so firm 2 wins the auction). The auction also ensures the welfare-maximizing allocation of the input increment if $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$ and either $m_2 > m_1$ (so firm 2 wins the auction) or $m_2 < m_1 - \frac{9t}{5} \mid D \mid$ (so firm 1 wins the auction). ■

As the proof of Proposition 2 indicates, when the input increment increases the value margins of the two firms at the same rate, welfare is highest when the input is allocated to the firm with the highest value margin, since this firm serves the most customers in equilibrium. (Recall equation (7).) Therefore, since an auction awards the increment to the firm with the highest value margin (recall Lemma 1), the auction ensures the welfare-maximizing allocation of the input increment in this case.

When an input increment would increase the value margin of, say, firm 1 more rapidly than it would increase the value margin of firm 2 (so $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$), welfare is highest when the input increment is awarded to firm 1 as long as firm 1's value margin is not too far below firm 2's value margin (i.e., as long as $m_1 \geq m_2 - \frac{9t}{5} \mid D \mid$). (Recall Lemma 2.) Therefore, when $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$, the auction (which awards the input increment to the firm with the highest value margin) will ensure the welfare-maximizing allocation of the input increment by awarding the increment to firm 1 when $m_1 > m_2$ and to firm 2 when $m_1 < m_2 - \frac{9t}{5} \mid D \mid$.

For completeness, Proposition 3 now specifies the conditions under which an unfettered auction will not generate the welfare-maximizing allocation of an input increment.

Proposition 3. *If* $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$ *and* $\Delta \in (\frac{9t}{5} D, 0)$*, then firm 1 will not win an auction for the input increment even though welfare would be higher if it did win the auction. In contrast, if* $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$ *and* $\Delta \in (0, \frac{9t}{5} D)$*, then firm 1 will win an auction for the input* *increment even though welfare would be higher if it did not win the auction.*

Proof. From Lemma 1, firm 1 wins the auction if $\Delta > 0$. From Lemma 2, welfare is highest when firm 1 wins the auction if $\Delta > \frac{9t}{5} D < 0$. Therefore, firm 1 does not win the auction when $\Delta \in (\frac{9t}{5} D, 0)$, even though welfare would be higher if firm 1 did win the auction.

Lemma 2 also indicates that welfare is highest when firm 1 wins the auction if $\Delta > \frac{9t}{5} D > 0$. Therefore, firm 1 wins the auction when $\Delta \in (0, \frac{9t}{5} D)$, even though welfare would be highest if firm 2 won the auction. ■

Proposition 3 implies that the private return a firm derives from securing more of a scarce input can exceed the corresponding social return when: (i) the input increases a rival's value margin more rapidly than it increases the firm's own value margin; and (ii) the firm's initial value margin moderately exceeds the rival's value margin, and so the firm serves a moderately larger number of customers than its rival. This moderately large customer base enhances the profitability of an increase in the firm's value margin and thereby motivates the firm to bid relatively aggressive for the input.

Figure 1 illustrates the conclusions drawn in Propositions 2 and 3. The Figure identifies the values of $\Delta \equiv m_1 - m_2$ and $D \equiv \frac{\frac{\partial m_2}{\partial k_2} - \frac{\partial m_1}{\partial k_1}}{\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}}$ for which an unfettered auction will, and will not, generate the welfare-maximizing allocation of the input increment. The auction will do so when the input increment would increase most rapidly the largest value margin (in the southeast and northwest quadrants of Figure 1) or the value margin that is the smallest by at least $\frac{9t}{5} \mid D \mid$ (in the identified regions of the northeast and southwest quadrants).

Corollary 2 helps to explain the slope of the line that passes through the point of intersection of the four quadrants in Figure 1, and thus the relative sizes of the regions in the northeast and southwest quadrants where an unfettered auction does, and does not, secure the welfare-maximizing allocation of the figure. The corollary indicates that the range of Δ values for which an auction fails to generate a welfare-maximizing allocation of the increment declines as consumers become less concerned with the horizontal dimensions of product quality (i.e., as t declines). Consumers of a wireless communications service, for example, may value particularly highly vertical dimensions of product quality like call clarity, the frequency of dropped calls, and geographic service coverage, but place relatively little value on horizontal dimensions of product quality (e.g., the proximity of a firm's showrooms to the city center).

Corollary 2. *The range of Δ values for which an unfettered auction fails to implement the welfare-maximizing allocation of an input increment declines as t declines.*

Proof. From Proposition 3, the set of Δ values for which an unfettered auction fails to implement the welfare-maximizing allocation of an input increment is: (i) $\Delta \in (\frac{9t}{5} D, 0)$ when $D < 0$; and (ii) $\Delta \in (0, \frac{9t}{5} D)$ when $D > 0$. Both sets of values contract as t declines because D does not vary with t (since $\frac{\partial m_i}{\partial k_i}$ is not a function of t). ■

As consumers place less weight on horizontal dimensions of product quality (i.e., as t declines), their innate preference for the product of a particular firm declines. Consequently, consumers become more inclined to purchase the product from the firm that delivers the highest value ($v(\cdot)$) at the lowest price. This change in consumer preferences enhances the congruence of the social value of the input and the values that the firms place on the input.[25]

Because unfettered auctions do not always generate welfare-maximizing allocations of scarce inputs, policymakers might conceivably attempt to modify auctions to enhance their performance in this regard. One modification that is employed in practice is to grant bid credits to certain potential bidders in order to encourage them to bid more aggressively for the scarce input. When firm i is awarded a bid credit of $b_i \in [\,0, 1\,)$, the firm is only required to pay $B_i\,[\,1 - b_i\,]$ for an input increment that it wins with a bid of B_i.[26] Proposition 4 characterizes the bid credit that ensures an auction will generate the welfare-maximizing

[25] The reductions in t considered in Corollary 2 are those for which Assumption 1 always holds. The variations do not include those that would cause the equilibrium to change from one in which both firms serve customers to one in which all customers purchase the product from the same firm.

[26] See Ayres and Cramton (1996), Cramton et al. (2011), and Athey et al. (2013), for example, for additional discussions and analyses of bid credits in auctions.

allocation of an input increment.

Proposition 4. *Suppose* $\frac{\partial m_i}{\partial k_i} > \frac{\partial m_j}{\partial k_j}$ *and* $m_j - m_i \in (0, \frac{9t}{5}|D|)$ *for* $j \neq i$, $i, j \in \{1, 2\}$. *Then an auction will ensure the welfare-maximizing allocation of the input increment if firm* i *is awarded a bid credit,* $b_i \in (0, 1)$, *that satisfies* $\frac{b_i}{2-b_i} = \frac{3}{5}|D|$.

Proof. From equations (14) and (15), the maximum amount that firm i will bid to secure the input increment when the firm receives bid credit $b_i \in (0, 1)$ is, for $j \neq i$, $i, j \in \{1, 2\}$:

$$\frac{B_i}{1-b_i} \equiv \frac{1}{1-b_i}\left[\frac{\partial \pi_i^*}{\partial k_i} - \frac{\partial \pi_j^*}{\partial k_j}\right] = \frac{1}{9t[1-b_i]}[3t + m_i - m_j]\left[\frac{\partial m_i}{\partial k_i} + \frac{\partial m_j}{\partial k_j}\right]. \tag{23}$$

Equations (14), (15), and (23) imply that when firm j receives no bid credit, firm i will outbid firm j for the input increment if:

$$\frac{1}{9t[1-b_i]}[3t + m_i - m_j]\left[\frac{\partial m_i}{\partial k_i} + \frac{\partial m_j}{\partial k_j}\right] > \frac{1}{9t}[3t + m_j - m_i]\left[\frac{\partial m_i}{\partial k_i} + \frac{\partial m_j}{\partial k_j}\right]$$

$$\Leftrightarrow \quad 3t + m_i - m_j > [1-b_i][3t + m_j - m_i] \quad \Leftrightarrow \quad m_i - m_j > 3t\left[\frac{b_i}{2-b_i}\right]. \tag{24}$$

Expression (24) implies that when $\frac{b_i}{2-b_i} = \frac{3}{5}|D|$, firm i will secure the input if:

$$m_i - m_j > 3t\left[\frac{3}{5}|D|\right] = \frac{9t}{5}|D|. \tag{25}$$

Expressions (22) and (25) imply that an auction with the identified bid credit will ensure the welfare-maximizing allocation of the input increment. ■

Proposition 4 reports that in order to ensure an auction will generate the welfare-maximizing allocation of an input increment, a bid credit can be awarded to the firm with a moderate value margin disadvantage when the input increment would increase its value margin more rapidly than it would increase the rival's value margin. The magnitude of the bid credit should increase with the extent to which the increment would increase the firm's value margin more than the rival's value margin, *ceteris paribus*.[27]

[27] Since $\frac{b_i}{2-b_i}$ is increasing in b_i, the bid credit that ensures the auction will generate the welfare-maximizing allocation of the input increment increases with $|D|$. Recall that $|D|$ increases linearly with $\left|\frac{\partial m_i}{\partial k_i} - \frac{\partial m_j}{\partial k_j}\right|$,

In practice, bid credits often are awarded to competitors that serve relatively few retail customers.[28] Proposition 4 identifies two ways in which such a policy can fail to ensure the welfare-maximizing allocation of an input increment. First, firms that serve the fewest retail customers may not be the firms whose value margins increase most rapidly as they acquire more of the input.[29] Second, even when an input increment would increase the value margin of a small competitor more than it would increase the value margin of a large competitor, welfare can be highest when the increment is awarded to the large competitor if its value margin (and thus its market share) sufficiently exceeds the value margin of the small competitor.

It should also be noted that substantial information about prevailing industry conditions is required to design the bid credits identified in Proposition 4.[30] One must know both the rates at which the input increases the value margins of the industry competitors and the difference between their value margins. In practice, this information can be very difficult, if not impossible, to obtain.

Before concluding, we note (in Proposition 5) that although welfare can decline as a firm acquires an input increment (recall Corollary 1), unfettered auctions will avoid welfare-reducing allocations of the increment.

Proposition 5. *An unfettered auction will never allocate an input increment in a manner that reduces welfare below the level that prevails in the absence of the increment.*

holding $\frac{\partial m_i}{\partial k_i} + \frac{\partial m_j}{\partial k_j}$ constant, for $j \neq i$, $i, j \in \{1, 2\}$. Observe that the magnitude of the bid credit in question does not vary with customer transportation costs, t.

[28] Cramton et al. (2011) observe that "The most common use of bidding credits has been in U.S. spectrum auctions, where they are granted to small businesses" (p. S171).

[29] To illustrate, suppliers of wireless communications service that serve many customers may face particularly severe spectrum shortages. Consequently, the value margins of these firms may increase relatively rapidly when they acquire additional spectrum.

[30] Policies other than bid credits might be considered to enhance auction performance. Cramton et al. (2011) discuss the potential roles of restrictions on auction participation and limits on the amount of the input that particular competitors can acquire in the auction. Milgrom (2004) and Bhattacharya et al. (2013), among others, note the potential benefits of limiting the number of bidders in settings where potential bidders must incur a cost in order to learn their valuations of the objects being auctioned.

Proof. Lemma 1 and equation (17) imply that if $\Delta > 0$, then firm 1 wins the auction and $\frac{\partial W^*}{\partial k_1} = \frac{\partial m_1}{\partial k_1}\left[\frac{1}{2} + \frac{5\,\Delta}{18\,t}\right] > 0$. Lemma 1 and equation (18) imply that if $\Delta < 0$, then firm 2 wins the auction and $\frac{\partial W^*}{\partial k_2} = \frac{\partial m_2}{\partial k_2}\left[\frac{1}{2} - \frac{5\,\Delta}{18\,t}\right] > 0$. If $\Delta = 0$, then equations (17) and (18) imply that $\frac{\partial W^*}{\partial k_i} = \frac{1}{2}\frac{\partial m_i}{\partial k_i} > 0$ for $i = 1, 2$. ■

The explanation for Proposition 5 is straightforward. An input increment will only reduce welfare if it is allocated to a firm with a particularly small value margin (as indicated in Corollary 1). Such a firm will never win an unfettered auction (as Lemma 1 reports).

5 Summary, Extensions and Conclusions

We have found that unfettered auctions typically ensure the welfare-maximizing allocation of a scarce input when the input increases the value margins of the competing suppliers symmetrically. Unfettered auctions also tend to perform relatively well in this regard when consumers value vertical dimensions of product quality particularly highly and place little weight on horizontal dimensions of product quality. Auctions can fail to allocate scarce inputs so as to maximize welfare, however. They do so, for instance, when the input increases relatively slowly the value margin of a firm that serves a moderately large share of the market or when the input increases relatively rapidly the value margin of a firm that serves a moderately small share of the market.

These findings suggest that the insights from the foreclosure literature may require some modification when considering settings where inputs have foreclosure value but cannot be employed to fully exclude competitors. We have found that when two firms compete in such a setting, the competitor with the larger market share will win an unfettered auction for an input increment, as the foreclosure literature might suggest. However, the resulting allocation of the input increment will increase welfare when the increment increases the value margin of the large firm at least as rapidly as it increases the value margin of the smaller firm. Therefore, when predicting the welfare implications of input allocations or when designing policies that affect the allocation of scarce inputs, it is important to assess both the levels of prevailing value margins (and associated market shares) and the relative rates at which firms' value margins change as they acquire more of the input. Policies that increase the access of small firms to scarce inputs can enhance welfare when additional units of the input increase the value margins of small firms relatively rapidly, but can otherwise reduce welfare.

This finding implies that considerable information is required to design welfare-enhancing policies (e.g., bid credits) that favor the access of certain industry competitors to scarce inputs. The requisite information typically is very difficult, if not impossible, for policymakers to obtain. Furthermore, policies that favor particular suppliers on the basis of endogenous characteristics can invite welfare-reducing strategic behavior. For instance, if favorable treatment is afforded to firms with small market shares and high marginal valuations of the input, then firms may find it profitable to reduce their market shares (perhaps by reducing the service quality they deliver to their customers) and to inflate their marginal valuations of the input (perhaps by installing relatively few substitute inputs).

Before concluding, we briefly discuss three extensions of our streamlined model. In the first extension, the unit transportation cost that consumers incur can differ across firms, reflecting the fact that one firm might enjoy a systematic inherent advantage on the horizontal dimension of product quality. Formally, let t_i denote the unit transportation cost that consumers incur when they purchase the product from firm $i \in \{1, 2\}$ in this *setting with asymmetric transportation costs*. The rate at which firm i's profit increases as it secures an input increment and thereby precludes its rival from securing the input in this setting is:[31]

$$\frac{2}{9\,[\,t_i + t_j\,]}\,[\,t_i + 2\,t_j + m_i - m_j\,]\left[\frac{\partial m_i}{\partial k_i} + \frac{\partial m_j}{\partial k_j}\right] \quad \text{for } j \neq i,\ i,\, j \in \{1, 2\}\,. \tag{26}$$

A comparison of expressions (14) and (26) reveals that the firm that enjoys a transportation cost advantage (i.e., firm i when $t_i < t_j$) values the input increment relatively highly. The high valuation arises because the firm's transportation cost advantage enables it to serve more customers and secure a higher price-cost margin in equilibrium, *ceteris paribus*. This

[31] See the Appendix for a formal proof of this conclusion and for the formal statements and proofs of the other conclusions that pertain to the setting with asymmetric transportation costs.

high valuation implies that a firm with a transportation cost advantage may secure the input increment auction even if it does not have the highest value margin.[32]

This difference aside, the key qualitative conclusions drawn above continue to hold in the setting with asymmetric transportation costs. In particular, if the input increment increases the value margins of the two competitors at the same rate (so $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2}$), then a firm will win an auction for the input increment if and only if welfare is highest when the firm secures the increment. In contrast, if the input increment increases relatively slowly the value margin of the firm with a moderate value margin advantage, then the firm may acquire an input increment even though welfare would increase if it did not do so.[33] Furthermore, as the unit transportation cost that consumers incur in purchasing from either of the two firms declines, the range of value margins for which an unfettered auction fails to produce the welfare-maximizing allocation of the input declines.

In the second extension of our model, firms can exert personally costly effort to reduce their production costs and enhance customer valuations of their products. The firms choose their effort supplied (simultaneously and independently) after the conclusion of the input auction, prior to setting prices.[34] Therefore, the input allocation affects welfare in this *setting with value-enhancing effort* in part by affecting the firms' subsequent effort choices.

This additional impact of the input allocation introduces some new conclusions. For example, the firm with the highest value margin does not necessarily acquire the input increment. The firm with the smaller value margin may acquire the input increment if,

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[32] Formally, firm i will secure the input increment if $m_i - m_j > \frac{1}{2}\left[t_i - t_j\right]$ for $j \neq i,\ \ i, j \in \{1, 2\}$. Perhaps surprisingly, an increase in t_i can increase firm i's profit in this setting by inducing the rival firm to increase its price. An increase in t_i also can increase firm i's marginal valuation of the scarce resource, and thereby increase the likelihood that firm i wins the auction for the input increment. Formally, $\frac{\partial}{\partial t_i}\left(\frac{\partial \pi_i^*}{\partial k_i} - \frac{\partial \pi_i^*}{\partial k_j}\right) > 0$ if $m_i < m_j - t_j$ for $j \neq i,\ \ i, j \in \{1, 2\}$.

[33] See Propositions 7 and 8 in the Appendix for the precise conditions under which an unfettered auction will not implement the welfare-maximizing allocation of an input increment in the setting with asymmetric transportation costs.

[34] This extension is in the spirit of the product repositioning that firms often undertake after they merge (e.g., Ghandi et al., 2008; Sweeting, 2010; Fan, 2013). In our case, "competitive repositioning" occurs after input increments have been allocated via auction.

by doing so, the firm anticipates a relatively pronounced reduction in its effort costs.[35] Furthermore, welfare can be highest when the input increment accrues to the firm that enjoys a moderate value margin advantage over a rival whose value margin would increase relatively rapidly if it received the input increment, holding constant the firms' effort levels. Welfare can be highest in this case if the additional input results in a relatively pronounced reduction in effort cost for the acquiring firm and if the lack of additional input for the rival firm does not raise its effort costs substantially.[36]

The third extension of our model admits competition among three firms, where firm $i \in \{1, 2, 3\}$ is located at position x_i on a circle of circumference 1. Consumers are distributed uniformly on this circumference and incur unit transportation cost t when traveling to purchase the product from any supplier. In this *triopoly setting*, the rate at which firm i's profit increases as it secures an input increment is:[37]

$$\frac{2}{25\,t}\left[t\,L_i + 2\,m_i - \sum_{j \neq i} m_j \right]\left[2\,\frac{\partial m_i}{\partial k_i} + \sum_{j \neq i} \alpha_{ij}\,\frac{\partial m_j}{\partial k_j} \right], \tag{27}$$

whereas the rate at which welfare increases as firm i secures the input increment is:

$$\begin{aligned} &\frac{1}{25\,t}\left[t\,(\,3 + 8\,L_i\,) + 16\,m_i - 8\sum_{j \neq i} m_j \right]\frac{\partial m_i}{\partial k_i} \\ &\quad - \sum_{h \neq i} \frac{\alpha_{ih}}{25\,t}\left[t\,(\,3 + 8\,L_h\,) + 16\,m_h - 8\sum_{j \neq h} m_j \right]\frac{\partial m_h}{\partial k_h} \quad \text{for } i,\, j,\, h \in \{1, 2, 3\}, \end{aligned} \tag{28}$$

where L_i denotes the sum of the distances between firm i and each of its rivals along the circle circumference, and α_{ij} denotes the probability that when firm i acquires the input increment at auction, it thereby precludes firm j from acquiring the increment.[38]

The triopoly setting introduces an important asymmetry that does not arise in the

[35] See expression (66) in the Appendix.

[36] A comparison of expressions (66) and (71) in the Appendix provides a formal specification of the conditions under which an unfettered auction will, and will not, ensure the welfare-maximizing allocation of the input in the setting with value-enhancing effort.

[37] For simplicity, the triopoly setting abstracts from any effort the firms might supply to reduce their production costs or enhance consumer valuation of their products.

[38] See Mayo and Sappington (2014) for a formal proof of these conclusions and additional conclusions pertaining to the triopoly setting.

duopoly setting analyzed in sections 2 – 4. When firm 1 acquires the input increment in the duopoly setting, it necessarily precludes its only rival, firm 2, from acquiring the increment. Firm 1 thereby derives: (i) a marginal use value that is proportional to its equilibrium price-cost margin ($p_1^* - c_1$) and $\frac{\partial m_1}{\partial k_1}$; and (ii) a marginal foreclosure value that is proportional to $p_1^* - c_1$ and $\frac{\partial m_2}{\partial k_2}$, for a combined marginal value that is proportional to $[\, p_1^* - c_1 \,] \left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2} \right]$. (Recall equation (14).) Similarly, firm 2 derives a marginal value from the input increment in the duopoly setting that is proportional to $[\, p_2^* - c_2 \,] \left[\frac{\partial m_2}{\partial k_2} + \frac{\partial m_1}{\partial k_1} \right]$. (Recall equation (15).) The symmetry in each rival's perceived gain from enhancing its own value margin and precluding its only rival from enhancing its value margin implies that the firm that secures the input increment at auction is the firm with the highest equilibrium price-cost margin, which, from equations (5) and (6), is the firm with the highest value margin. (Recall Lemma 1.)

A corresponding symmetry does not arise in the triopoly setting. When firm i secures the input increment at auction in this setting, the firm derives no foreclosure value with regard to the rival that would not have obtained the input even if firm i had not secured the increment. The resulting asymmetry in the sum of marginal use and foreclosure values implies that the allocation of the input in the triopoly setting typically will depend upon both the relative magnitudes of the firms' value margins and the relative rates at which these value margins increase as a firm acquires more of the critical input.[39]

To illustrate this more general conclusion, suppose the firms are equally spaced around the circle,[40] firm 1's value margin exceeds the value margins of its symmetric rivals (i.e., $m_1 > m_2 = m_3$), and firm 1's value margin increases more slowly as it secures more of the input than do the value margins of the rivals (i.e., $\frac{\partial m_1}{\partial k_1} < \frac{\partial m_2}{\partial k_2} = \frac{\partial m_3}{\partial k_3}$). When $m_1 - m_2$ is large relative to $\frac{\partial m_2}{\partial k_2} - \frac{\partial m_1}{\partial k_1}$, firm 1 will value the input increment most highly, and so will

[39] This asymmetry is apparent in the second term in square brackets in expression (27). The coefficient of 2 on the $\frac{\partial m_i}{\partial k_i}$ term reflects the marginal use value that firm i derives from the input increment in competing against its two rivals. The α_{ij} coefficients reflect the fact that firm i typically derives a foreclosure value from securing the input increment only with respect to one of its rivals.

[40] This equal spacing assumption abstracts from systematic advantages and disadvantages with regard to the horizontal dimension of product quality, such as those that can be analyzed in the setting with asymmetric transportation costs.

be the firm whose access to the increment is foreclosed should firm 2 or firm 3 submit the highest bid for the input increment (so $\alpha_{21} = \alpha_{31} = 1$ and $\alpha_{23} = \alpha_{32} = 0$). Expression (27) implies that firm 1 will win the auction for the input increment under these conditions if $\frac{\frac{\partial m_1}{\partial k_1}}{\frac{\partial m_2}{\partial k_2}} > \frac{5t-12[m_1-m_2]}{5t+15[m_1-m_2]} \equiv r_f$. Therefore, the identity of the firm that wins the auction depends upon both relative margin values and relative rates at which margin values vary with the input. Expression (28) implies that welfare is highest when firm 1 acquires the increment under the specified conditions if $\frac{\frac{\partial m_1}{\partial k_1}}{\frac{\partial m_2}{\partial k_2}} > \frac{25t-24[m_1-m_2]}{25t+48[m_1-m_2]} \equiv r_w$. It is readily verified that $r_w > r_f$. Therefore, as in the duopoly setting, firm 1's relatively high value margin may lead it to acquire the input increment when $\frac{\partial m_1}{\partial k_1}$ is low relative to $\frac{\partial m_2}{\partial k_2}$ even though welfare would be higher if firm 2 or firm 3 acquired the input increment.

When the increment increases the value margins of the three competitors symmetrically (so $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2} = \frac{\partial m_3}{\partial k_3}$) in the triopoly setting, a firm will acquire the input increment in an unfettered auction precisely when welfare is highest if it does so. Thus, the natural counterpart to conclusion (i) in Proposition 2 holds in the triopoly setting.

In closing, we note that our analysis has focused on the extent to which auctions ensure the welfare-maximizing allocation of scarce inputs. In practice, policymakers may pursue objectives other than welfare maximization. For example, policymakers may value the revenue derived from the sale of inputs and/or seek to promote industry participation by small businesses and minority business owners.[41] Detailed investigation of these other objectives awaits further research. Future research should also consider incomplete information, alternative demand formulations (including settings where individual consumers may purchase more than a single unit of the product), and the incentives for free-riding that can arise in settings with more than two industry suppliers.[42]

[41] Cramton et al. (2011, p. S169) report that they "consider the primary goal of the regulator to be economic efficiency". However, the authors also note other goals of spectrum auctions, including revenue generation.

[42] McAfee (1998), for example, shows that when multiple industry suppliers do not face capacity constraints, each supplier may be unable to capture the full increase in the profit of the unconstrained suppliers that arises when additional capacity is withheld from capacity-constrained suppliers. The relatively small foreclosure value that each unconstrained supplier anticipates from securing additional capacity at auction implies that the capacity-constrained firms (e.g., new industry entrants) may secure auctioned capacity increments.

Appendix

This Appendix presents the analysis that underlies the findings reported in section 5. The analysis in part A below also provides more detailed proofs of the formulae presented in section 2.

A. The Setting with Asymmetric Transportation Costs.

Let t_i denote the unit transportation cost that consumers incur when they travel to purchase the product from firm $i \in \{1, 2\}$. In this setting, the (interior) location of the consumer who is indifferent between purchasing the product from the two firms is determined by:

$$v_1 - p_1 - t_1 x \;=\; v_2 - p_2 - t_2 [1 - x] \quad \Rightarrow \quad [t_1 + t_2] x \;=\; t_2 + v_1 - p_1 - v_2 + p_2$$

$$\Rightarrow \quad \widehat{x} \;=\; \frac{1}{t_1 + t_2} [t_2 + v_1 - v_2 + p_2 - p_1] \tag{29}$$

$$\Rightarrow \quad 1 - \widehat{x} \;=\; \frac{1}{t_1 + t_2} [t_1 + v_2 - v_1 + p_1 - p_2]. \tag{30}$$

(29) implies that at an interior solution with full-market coverage (in which all consumers buy a unit of the product and each firm serves some customers), firm 1's profit is:

$$\pi_1 \;=\; [p_1 - c_1] \, \widehat{x} \;=\; \frac{1}{t_1 + t_2} [p_1 - c_1] [t_2 + v_1 - v_2 + p_2 - p_1]. \tag{31}$$

(31) implies that firm 1's profit-maximizing price (at an interior solution) is determined by:

$$\frac{\partial \pi_1}{\partial p_1} \stackrel{s}{=} -[p_1 - c_1] + t_2 + v_1 - v_2 + p_2 - p_1 \;=\; 0$$

$$\Rightarrow \quad p_1 \;=\; \frac{1}{2} [t_2 + c_1 + v_1 - v_2 + p_2]. \tag{32}$$

(30) implies that firm 2's profit is:

$$\pi_2 \;=\; [p_2 - c_2][1 - \widehat{x}] \;=\; \frac{1}{t_1 + t_2} [p_2 - c_2] [t_1 + v_2 - v_1 + p_1 - p_2]. \tag{33}$$

(33) implies that firm 2's profit-maximizing price is determined by:

$$\frac{\partial \pi_2}{\partial p_2} \stackrel{s}{=} -[p_2 - c_2] + t_1 + v_2 - v_1 + p_1 - p_2 \;=\; 0$$

$$\Rightarrow \quad p_2 \;=\; \frac{1}{2} [t_1 + c_2 + v_2 - v_1 + p_1]. \tag{34}$$

Solving (32) and (34) simultaneously provides:

$$p_1 \;=\; \frac{1}{2} [t_2 + c_1 + v_1 - v_2] + \frac{1}{4} [t_1 + c_2 + v_2 - v_1 + p_1]$$

$$\Rightarrow \quad \frac{3}{4}\,p_1 \;=\; \frac{1}{4}\,[\,2\,t_2 + 2\,c_1 + 2\,v_1 - 2\,v_2 + t_1 + c_2 + v_2 - v_1\,]$$

$$\Rightarrow \quad p_1^* \;=\; \frac{1}{3}\,[\,t_1 + 2\,t_2 + v_1 - v_2 + 2\,c_1 + c_2\,] \tag{35}$$

$$\Rightarrow \quad p_1^* - c_1 \;=\; \frac{1}{3}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]\,. \tag{36}$$

Substituting (35) into (34) provides:

$$p_2^* \;=\; \frac{1}{2}\,[\,t_1 + c_2 + v_2 - v_1\,] + \frac{1}{6}\,[\,t_1 + 2\,t_2 + v_1 - v_2 + 2\,c_1 + c_2\,]$$

$$=\; \frac{1}{6}\,[\,3\,t_1 + 3\,c_2 + 3\,v_2 - 3\,v_1 + t_1 + 2\,t_2 + v_1 - v_2 + 2\,c_1 + c_2\,]$$

$$\Rightarrow \quad p_2^* \;=\; \frac{1}{3}\,[\,t_2 + 2\,t_1 + v_2 - v_1 + 2\,c_2 + c_1\,] \tag{37}$$

$$\Rightarrow \quad p_2^* - c_2 \;=\; \frac{1}{3}\,[\,t_2 + 2\,t_1 + m_2 - m_1\,]\,. \tag{38}$$

(35) and (37) imply:

$$p_2^* - p_1^* \;=\; \frac{1}{3}\,[\,t_1 - t_2 + 2\,v_2 - 2\,v_1 + c_2 - c_1\,]\,. \tag{39}$$

(29) and (39) imply that firm 1's equilibrium output is:

$$x_1^* \;=\; \frac{1}{t_1 + t_2}\left[\,t_2 + v_1 - v_2 + \frac{1}{3}\,(\,t_1 - t_2 + 2\,v_2 - 2\,v_1 + c_2 - c_1\,)\right]$$

$$=\; \frac{1}{t_1 + t_2}\left[\,\frac{1}{3}\,t_1 + \frac{2}{3}\,t_2 + \frac{1}{3}\,v_1 - \frac{1}{3}\,v_2 + \frac{1}{3}\,c_2 - \frac{1}{3}\,c_1\right]$$

$$=\; \frac{1}{3\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]\,. \tag{40}$$

(40) implies that firm 2's equilibrium output is:

$$x_2^* \;=\; 1 - x_1^* \;=\; \frac{1}{3\,[\,t_1 + t_2\,]}\,[\,t_2 + 2\,t_1 + m_2 - m_1\,]\,. \tag{41}$$

To ensure $x_1^* \in (0,1)$ and $x_2^* \in (0,1)$, Assumption 2 is presumed to hold throughout the ensuing analysis.

Assumption 2. $-\,(\,t_1 + 2\,t_2\,) \,<\, m_1 - m_2 \,<\, t_2 + 2\,t_1\,.$

From (36) and (40), firm 1's equilibrium profit (given v_1, v_2, c_1, and c_2) is:

$$\pi_1^* \;=\; [\,p_1^* - c_1\,]\,x_1^* \;=\; \frac{1}{9\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]^2 \tag{42}$$

$$\Rightarrow \quad \frac{\partial \pi_1^*}{\partial k_1} \;=\; \frac{2}{9\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]\,\frac{\partial m_1}{\partial k_1} \quad \text{and}$$

$$\frac{\partial \pi_1^*}{\partial k_2} \;=\; -\,\frac{2}{9\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]\,\frac{\partial m_2}{\partial k_2}\,. \tag{43}$$

Similarly, from (38) and (41):

$$\pi_2^* \;=\; [\,p_2^* - c_2\,]\,x_2^* \;=\; \frac{1}{9\,[\,t_1 + t_2\,]}\,[\,t_2 + 2\,t_1 + m_2 - m_1\,]^2 \tag{44}$$

$$\Rightarrow \quad \frac{\partial \pi_2^*}{\partial k_2} \;=\; \frac{2}{9\,[\,t_1 + t_2\,]}\,[\,t_2 + 2\,t_1 + m_2 - m_1\,]\,\frac{\partial m_2}{\partial k_2} \quad \text{and}$$

$$\frac{\partial \pi_2^*}{\partial k_1} \;=\; -\,\frac{2}{9\,[\,t_1 + t_2\,]}\,[\,t_2 + 2\,t_1 + m_2 - m_1\,]\,\frac{\partial m_1}{\partial k_1}\,. \tag{45}$$

(43) implies that the maximum amount firm 1 will pay to secure an increment in the input rather than let firm 2 secure the increment is proportional to:

$$B_1 \;\equiv\; \frac{\partial \pi_1^*}{\partial k_1} - \frac{\partial \pi_1^*}{\partial k_2} \;=\; \frac{2}{9\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]. \tag{46}$$

Similarly, (45) implies that the maximum amount firm 2 will pay to secure an increment in k rather than let firm 1 secure the increment is proportional to:

$$B_2 \;\equiv\; \frac{\partial \pi_2^*}{\partial k_2} - \frac{\partial \pi_2^*}{\partial k_1} \;=\; \frac{2}{9\,[\,t_1 + t_2\,]}\,[\,t_2 + 2\,t_1 + m_2 - m_1\,]\left[\frac{\partial m_2}{\partial k_2} + \frac{\partial m_1}{\partial k_1}\right]. \tag{47}$$

(46) and (47) imply that firm 1 will win an auction for the input increment if:

$$B_1 \;>\; B_2 \quad \Leftrightarrow \quad t_1 + 2\,t_2 + m_1 - m_2 \;>\; t_2 + 2\,t_1 + m_2 - m_1$$

$$\Leftrightarrow \quad 2\,m_1 - 2\,m_2 \;>\; t_1 - t_2 \quad \Leftrightarrow \quad \Delta \;\equiv\; m_1 - m_2 \;>\; \frac{1}{2}\,[\,t_1 - t_2\,]\,. \tag{48}$$

Let $\pi^* \equiv \pi_1^* + \pi_2^*$. Then, since $\Delta \equiv m_1 - m_2$, (42) and (44) imply:

$$\pi^* \;=\; \frac{1}{9\,[\,t_1 + t_2\,]}\,\{\,[\,t_1 + 2\,t_2 + \Delta\,]^2 + [\,t_2 + 2\,t_1 - \Delta\,]^2\,\}\,. \tag{49}$$

Differentiating (49) provides:

$$\frac{\partial \pi^*}{\partial k_1} = \frac{2}{9\,[\,t_1 + t_2\,]}\,[\,(\,t_1 + 2\,t_2 + \Delta\,) - (\,t_2 + 2\,t_1 - \Delta\,)\,]\,\frac{\partial m_1}{\partial k_1}$$

$$= \frac{2\,[\,2\,\Delta + t_2 - t_1\,]}{9\,[\,t_1 + t_2\,]}\,\frac{\partial m_1}{\partial k_1}\,. \tag{50}$$

Similarly:

$$\frac{\partial \pi^*}{\partial k_2} = \frac{2}{9\,[\,t_1 + t_2\,]}\,[\,-(\,t_1 + 2\,t_2 + \Delta\,) + (\,t_2 + 2\,t_1 - \Delta\,)\,]\,\frac{\partial m_2}{\partial k_2}$$

$$= -\,\frac{2\,[\,2\,\Delta + t_2 - t_1\,]}{9\,[\,t_1 + t_2\,]}\,\frac{\partial m_2}{\partial k_2}\,. \tag{51}$$

(50) and (51) imply that the rate at which industry profit increases as the input is transferred from firm 2 to firm 1 (or, in an auction, is awarded to firm 1 rather than firm 2) is:

$$\frac{\partial \pi^*}{\partial k_1} - \frac{\partial \pi^*}{\partial k_2} = \frac{2\,[\,2\,\Delta + t_2 - t_1\,]}{9\,[\,t_1 + t_2\,]}\left[\,\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\,\right]. \tag{52}$$

(35), (37), (40), and (41) imply that equilibrium consumer surplus is:

$$CS^* = \int_0^{x_1^*} [\,v_1 - p_1^* - t_1\,x\,]\,dx + \int_{x_1^*}^{1} [\,v_2 - p_2^* - t_2\,(1-x)\,]\,dx$$

$$= [\,v_1 - p_1^*\,]\,x_1^* - t_1\,\frac{1}{2}\,x^2\Big|_0^{x_1^*} + [\,v_2 - p_2^*\,]\,[\,1 - x_1^*\,] - t_2\;x\Big|_{x_1^*}^{1} + t_2\,\frac{1}{2}\,x^2\Big|_{x_1^*}^{1}$$

$$= [\,v_1 - p_1^*\,]\,x_1^* - \frac{t_1}{2}\,(x_1^*)^2 + [\,v_2 - p_2^*\,]\,[\,1 - x_1^*\,] - t_2\,[\,1 - x_1^*\,] + \frac{t_2}{2}\,[\,1 - (x_1^*)^2\,]$$

$$= [\,v_1 - p_1^*\,]\,x_1^* + [\,v_2 - p_2^*\,]\,[\,1 - x_1^*\,] - \frac{t_2}{2} + t_2\,x_1^* - \frac{t_1 + t_2}{2}\,(x_1^*)^2$$

$$= [\,v_1 - p_1^*\,]\,x_1^* + [\,v_2 - p_2^*\,]\,[\,1 - x_1^*\,] - \frac{t_2}{2} + x_1^*\left[\,t_2 - \left(\frac{t_1 + t_2}{2}\right)x_1^*\,\right]$$

$$= \left[\,v_1 - \frac{1}{3}\,(\,t_1 + 2\,t_2 + v_1 - v_2 + 2\,c_1 + c_2\,)\,\right]\frac{1}{3\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]$$

$$+ \left[\,v_2 - \frac{1}{3}\,(\,t_2 + 2\,t_1 + v_2 - v_1 + 2\,c_2 + c_1\,)\,\right]\frac{1}{3\,[\,t_1 + t_2\,]}\,[\,t_2 + 2\,t_1 + m_2 - m_1\,] - \frac{t_2}{2}$$

$$+ \frac{1}{3\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + m_1 - m_2\,]\left[\,t_2 - \left(\frac{t_1 + t_2}{2}\right)\frac{t_1 + 2\,t_2 + m_1 - m_2}{3\,[\,t_1 + t_2\,]}\,\right]$$

$$= \frac{1}{3\,[\,t_1 + t_2\,]} \left[v_1 - \frac{1}{3}\,(\,t_1 + 2\,t_2 + \Delta + 3\,c_1\,) \right] [\,t_1 + 2\,t_2 + \Delta\,]$$

$$+ \frac{1}{3\,[\,t_1 + t_2\,]} \left[v_2 - \frac{1}{3}\,(\,t_2 + 2\,t_1 - \Delta + 3\,c_2\,) \right] [\,t_2 + 2\,t_1 - \Delta\,] - \frac{t_2}{2}$$

$$+ \frac{1}{3\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + \Delta\,] \left[t_2 - \frac{t_1 + 2\,t_2 + \Delta}{6} \right]$$

$$= \frac{1}{9\,[\,t_1 + t_2\,]}\,[\,3\,v_1 - (\,t_1 + 2\,t_2 + \Delta + 3\,c_1\,)\,]\,[\,t_1 + 2\,t_2 + \Delta\,]$$

$$+ \frac{1}{9\,[\,t_1 + t_2\,]}\,[\,3\,v_2 - (\,t_2 + 2\,t_1 - \Delta + 3\,c_2\,)\,]\,[\,t_2 + 2\,t_1 - \Delta\,] - \frac{t_2}{2}$$

$$+ \frac{1}{9\,[\,t_1 + t_2\,]}\,[\,t_1 + 2\,t_2 + \Delta\,]\,\frac{1}{2}\,[\,6\,t_2 - (\,t_1 + 2\,t_2 + \Delta\,)\,]$$

$$= -\frac{t_2}{2} + \frac{1}{9\,[\,t_1 + t_2\,]}\,\{\,[\,3\,m_1 - (\,t_1 + 2\,t_2 + \Delta\,)\,]\,[\,t_1 + 2\,t_2 + \Delta\,]$$

$$+ [\,3\,m_2 - (\,t_2 + 2\,t_1 - \Delta\,)\,]\,[\,t_2 + 2\,t_1 - \Delta\,]$$

$$+ \frac{1}{2}\,[\,t_1 + 2\,t_2 + \Delta\,]\,[\,6\,t_2 - (\,t_1 + 2\,t_2 + \Delta\,)\,]\,\}$$

$$= -\frac{t_2}{2} + \frac{1}{9\,[\,t_1 + t_2\,]}\,\{\,3\,[\,m_1 + t_2\,]\,[\,t_1 + 2\,t_2 + \Delta\,] + 3\,m_2\,[\,t_2 + 2\,t_1 - \Delta\,]$$

$$- \frac{3}{2}\,[\,t_1 + 2\,t_2 + \Delta\,]^2 - [\,t_2 + 2\,t_1 - \Delta\,]^2\,\}\,. \tag{53}$$

(49) and (53) imply that equilibrium welfare is:

$$W^* = CS^* + \pi^* = -\frac{t_2}{2} + \frac{1}{9\,[\,t_1 + t_2\,]}\,\{\,3\,[\,m_1 + t_2\,]\,[\,t_1 + 2\,t_2 + \Delta\,]$$

$$+ 3\,m_2\,[\,t_2 + 2\,t_1 - \Delta\,] - \frac{1}{2}\,[\,t_1 + 2\,t_2 + \Delta\,]^2\,\}\,. \tag{54}$$

Differentiating (54) reveals that the rate at which equilibrium welfare increases as firm 1 receives more of the input is:

$$\frac{\partial W^*}{\partial k_1} = \frac{1}{9\,[\,t_1 + t_2\,]}\,\frac{\partial m_1}{\partial k_1}\,\{\,3\,m_1 + 3\,t_2 + 3\,[\,t_1 + 2\,t_2 + \Delta\,] - 3\,m_2 - [\,t_1 + 2\,t_2 + \Delta\,]\,\}$$

$$= \frac{1}{9\,[\,t_1 + t_2\,]}\,\frac{\partial m_1}{\partial k_1}\,[\,7\,t_2 + 2\,t_1 + 3\,(\,m_1 - m_2\,) + 2\,\Delta\,]$$

$$= \frac{\partial m_1}{\partial k_1} \left[\frac{7\,t_2 + 2\,t_1 + 5\,\Delta}{9\,(\,t_1 + t_2\,)} \right] \tag{55}$$

Similarly, the rate at which equilibrium welfare increases as firm 2 receives more of the input is:

$$\begin{aligned}
\frac{\partial W^*}{\partial k_2} &= \frac{1}{9\,[\,t_1+t_2\,]}\,\frac{\partial m_2}{\partial k_2}\,\{\,-3\,m_1-3\,t_2\,+3\,m_2+3\,[\,t_2+2\,t_1-\Delta\,]+[\,t_1+2\,t_2+\Delta\,]\,\}\\
&= \frac{1}{9\,[\,t_1+t_2\,]}\,\frac{\partial m_2}{\partial k_2}\,[\,7\,t_1+2\,t_2-3\,(\,m_1-m_2\,)-2\,\Delta\,]\\
&= \frac{\partial m_2}{\partial k_2}\left[\frac{7\,t_1+2\,t_2-5\,\Delta}{9\,(\,t_1+t_2\,)}\right]. \qquad (56)
\end{aligned}$$

(55) and (56) imply that the rate at which welfare increases as firm 1 secures the input increment and thereby precludes firm 2 from doing so is:

$$\begin{aligned}
G_1 \equiv \frac{\partial W^*}{\partial k_1}-\frac{\partial W^*}{\partial k_2} &= \frac{1}{9\,[\,t_1+t_2\,]}\left[\frac{\partial m_1}{\partial k_1}\,(\,7\,t_2+2\,t_1\,)-\frac{\partial m_2}{\partial k_2}\,(\,7\,t_1+2\,t_2\,)\right]\\
&\quad+\frac{5\,\Delta}{9\,[\,t_1+t_2\,]}\left[\frac{\partial m_1}{\partial k_1}+\frac{\partial m_2}{\partial k_2}\right]. \qquad (57)
\end{aligned}$$

The rate at which welfare increases as firm 2 secures the input increment and thereby precludes firm 1 from doing so is:

$$G_2 \equiv \frac{\partial W^*}{\partial k_2}-\frac{\partial W^*}{\partial k_1} = -\,G_1\,. \qquad (58)$$

(57) and (58) imply that welfare increases more rapidly as firm 1, rather than firm 2, secures the input increment if:

$$\begin{aligned}
& G_1 > G_2 \;\Leftrightarrow\; G_1 > -G_1 \;\Leftrightarrow\; G_1 > 0\\
\Leftrightarrow\;& \frac{\partial m_1}{\partial k_1}\,[\,7\,t_2+2\,t_1\,]-\frac{\partial m_2}{\partial k_2}\,[\,7\,t_1+2\,t_2\,]+5\,\Delta\left[\frac{\partial m_1}{\partial k_1}+\frac{\partial m_2}{\partial k_2}\right] > 0\\
\Leftrightarrow\;& 5\,\Delta\left[\frac{\partial m_1}{\partial k_1}+\frac{\partial m_2}{\partial k_2}\right] > \frac{\partial m_2}{\partial k_2}\,[\,7\,t_1+2\,t_2\,]-\frac{\partial m_1}{\partial k_1}\,[\,7\,t_2+2\,t_1\,]\\
\Leftrightarrow\;& \Delta \equiv m_1-m_2 > \widetilde{D} \qquad (59)
\end{aligned}$$

$$\text{where}\quad \widetilde{D} \equiv \left[\frac{\frac{\partial m_2}{\partial k_2}\,[\,7\,t_1+2\,t_2\,]-\frac{\partial m_1}{\partial k_1}\,[\,7\,t_2+2\,t_1\,]}{5\left(\frac{\partial m_1}{\partial k_1}+\frac{\partial m_2}{\partial k_2}\right)}\right]. \qquad (60)$$

Before stating in this setting with asymmetric transportation costs the counterparts to the formal conclusions in section 4, we characterize the impact of changes in customer transportation costs on equilibrium profits, welfare, and marginal valuations of the scarce input.

Observation 1. $\frac{\partial \pi_2^*}{\partial t_1} > 0$.

Proof. From (44):

$$\frac{\partial \pi_2^*}{\partial t_1} \stackrel{s}{=} 2\,[\,t_1 + t_2\,]\,2\,[\,t_2 + 2\,t_1 - \Delta\,] - [\,t_2 + 2\,t_1 - \Delta\,]^2$$

$$\stackrel{s}{=} 4\,[\,t_1 + t_2\,] - [\,t_2 + 2\,t_1 - \Delta\,] \;=\; 2\,t_1 + 3\,t_2 + \Delta \;>\; 0\,.$$

The second "$\stackrel{s}{=}$" sign and the inequality here reflect Assumption 2, which ensures $t_2 + 2\,t_1 - \Delta > 0$ and $2\,t_1 + 3\,t_2 + \Delta \; > \; t_1 + 2\,t_2 + \Delta \; > \; 0$. ■

Observation 1 implies that a firm always benefits from an increase in the transportation cost that consumers incur in purchasing the rival's product.

Observation 2. $\frac{\partial \pi_1^*}{\partial t_1} \gtreqless 0$ *as* $m_1 - m_2 \lesseqgtr t_1$.

Proof. From (42):

$$\frac{\partial \pi_1^*}{\partial t_1} \stackrel{s}{=} 2\,[\,t_1 + t_2\,]\,[\,t_1 + 2\,t_2 + \Delta\,] - [\,t_1 + 2\,t_2 + \Delta\,]^2$$

$$\stackrel{s}{=} 2\,[\,t_1 + t_2\,] - [\,t_1 + 2\,t_2 + \Delta\,] \;=\; t_1 - \Delta \;\gtreqless\; 0 \;\;\Leftrightarrow\;\; \Delta \;\lesseqgtr\; t_1\,.$$

The second "$\stackrel{s}{=}$" sign here reflects Assumption 2. ■

Observation 2 implies that firm 1's profit will increase as consumers find it more costly to purchase from the firm (i.e., as t_1 increases) as long as firm 1's value margin is not too much larger than firm 2's value margin. Firm 1 gains in this case in part because firm 2 increases its price as it faces a weaker rival. (Observe from (37) and (39) that p_2^* and $p_2^* - p_1^*$ increase as t_1 increases.)

Observation 3. $\frac{\partial \pi^*}{\partial t_1} > 0$ *if* Δ *is sufficiently close to* 0.

Proof. From (49):

$$\frac{\partial \pi^*}{\partial t_1} \stackrel{s}{=} 2\,[\,t_1 + t_2\,]\,[\,t_1 + 2\,t_2 + \Delta + 2\,t_2 + 4\,t_1 - 2\,\Delta\,] - [\,t_1 + 2\,t_2 + \Delta\,]^2 - [\,t_2 + 2\,t_1 - \Delta\,]^2$$

$$= 2\,[\,t_1 + t_2\,]\,[\,t_1 + 2\,t_2\,] + 4\,[\,t_1 + t_2\,]\,[\,t_2 + 2\,t_1\,] - 2\,\Delta\,[\,t_1 + t_2\,] - [\,t_1 + 2\,t_2\,]^2$$
$$- 2\,\Delta\,[\,t_1 + 2\,t_2\,] - \Delta^2 - [\,t_2 + 2\,t_1\,]^2 + 2\,\Delta\,[\,t_2 + 2\,t_1\,] - \Delta^2$$

$$= [\,t_1 + 2\,t_2\,]\,[\,2\,t_1 + 2\,t_2 - t_1 - 2\,t_2\,] + [\,t_2 + 2\,t_1\,]\,[\,4\,t_1 + 4\,t_2 - t_2 - 2\,t_1\,]$$
$$- 2\,\Delta\,[\,t_1 + t_2 + t_1 + 2\,t_2 - t_2 - 2\,t_1\,] - 2\,\Delta^2$$

$$= \; [\, t_1 + 2\, t_2 \,]\, t_1 + [\, t_2 + 2\, t_1 \,]\, [\, 2\, t_1 + 3\, t_2 \,] - 4\, \Delta\, t_2 - 2\, \Delta^2 \, .$$

It is apparent that this expression is strictly positive when Δ is sufficiently close to 0. ■

Observation 4. $\frac{\partial B_1}{\partial t_1} = \frac{\partial}{\partial t_1} \left(\frac{\partial \pi_1^*}{\partial k_1} - \frac{\partial \pi_1^*}{\partial k_2} \right) \gtreqless 0$ *as* $m_1 - m_2 \lesseqgtr -t_2$.

Proof. From (46):

$$\frac{\partial B_1}{\partial t_1} \overset{s}{=} t_1 + t_2 - [\, t_1 + 2\, t_2 + \Delta \,] \; = \; - \, (\, t_2 + \Delta \,) \; \gtreqless \; 0 \; \Leftrightarrow \; \Delta \; \lesseqgtr \; - \, t_2 \, . \quad ■$$

Observation 4 indicates that an increase in t_1 increases firm 1's marginal valuation of the input when firm 1's value margin is sufficiently small relative to firm 2's value margin. This conclusion reflects the fact that firm 1's market share increases as t_1 increases in this case.

Observation 5. $\frac{\partial (B_1 - B_2)}{\partial t_1} \gtreqless 0$ *as* $m_1 - m_2 \lesseqgtr -t_2$.

Proof. From (46) and (47):

$$\begin{aligned} \frac{\partial\, (B_1 - B_2)}{\partial t_1} \; &\overset{s}{=} \; \frac{\partial}{\partial t_1} \left(\frac{t_1 + 2\, t_2 + m_1 - m_2 - [\, t_2 + 2\, t_1 + m_2 - m_1 \,]}{t_1 + t_2} \right) \\ &\overset{s}{=} \; \frac{\partial}{\partial t_1} \left(\frac{t_2 - t_1 + 2\, m_1 - 2\, m_2}{t_1 + t_2} \right) \; \overset{s}{=} \; - \, [\, t_1 + t_2 \,] - [\, t_2 - t_1 + 2\, \Delta \,] \\ &= \; - \, 2\, [\, t_2 + \Delta \,] \; \gtreqless \; 0 \; \Leftrightarrow \; \Delta \; \equiv \; m_1 - m_2 \; \lesseqgtr \; - \, t_2 \, . \quad ■ \end{aligned}$$

Observation 5 indicates that an increase in t_1 increases firm 1's differential marginal valuation of the input when firm 1's value margin is sufficiently small relative to firm 2's value margin. Consequently, firm 1 can become more likely to win the auction for the input increment as it becomes more costly for firm 1's customers to purchase the product from firm 1.

Observation 6. $\frac{\partial W^*}{\partial t_1} < 0$.

Proof. From (54):

$$\begin{aligned} \frac{\partial W^*}{\partial t_1} \; &\overset{s}{=} \; [\, t_1 + t_2 \,]\, [\, 3\, (\, m_1 + t_2 \,) + 6\, m_2 - (\, t_1 + 2\, t_2 + \Delta \,) \,] - 3\, [\, m_1 + t_2 \,]\, [\, t_1 + 2\, t_2 + \Delta \,] \\ &\qquad - \, 3\, m_2\, [\, t_2 + 2\, t_1 - \Delta \,] + \frac{1}{2}\, [\, t_1 + 2\, t_2 + \Delta \,]^2 \\ &= \; - \, 3\, [\, m_1 + t_2 \,]\, [\, t_1 + 2\, t_2 + \Delta - t_1 - t_2 \,] + 3\, m_2\, [\, 2\, t_1 + 2\, t_2 - t_2 - 2\, t_1 + \Delta \,] \\ &\qquad - \, \frac{1}{2}\, [\, t_1 + 2\, t_2 + \Delta \,]\, [\, 2\, t_1 + 2\, t_2 - t_1 - 2\, t_2 - \Delta \,] \end{aligned}$$

$$= \; -3\,[\,m_1 + t_2\,]\,[\,t_2 + \Delta\,] + 3\,m_2\,[\,t_2 + \Delta\,] - \frac{1}{2}\,[\,t_1 + 2\,t_2 + \Delta\,]\,[\,t_1 - \Delta\,]$$

$$= \; -3\,[\,t_2 + \Delta\,]\,[\,m_1 - m_2 + t_2\,] - \frac{1}{2}\,[\,t_1 + 2\,t_2 + \Delta\,]\,[\,t_1 - \Delta\,]$$

$$= \; -3\,[\,t_2 + \Delta\,]\,[\,t_2 + \Delta\,] - \frac{1}{2}\left[\,t_1^2 + 2\,t_1\,t_2 + t_1\,\Delta - t_1\,\Delta - 2\,t_2\,\Delta - \Delta^2\,\right]$$

$$= \; -3\left[\,t_2^2 + 2\,t_2\,\Delta + \Delta^2\,\right] - \frac{1}{2}\left[\,t_1^2 + 2\,t_1\,t_2 - 2\,t_2\,\Delta - \Delta^2\,\right]$$

$$= \; -3\,t_2^2 - \frac{1}{2}\,t_1^2 - t_1\,t_2 - 5\,t_2\,\Delta - \frac{5}{2}\,\Delta^2 \; = \; -\frac{1}{2}\left[\,5\,\Delta^2 + 10\,t_2\,\Delta + r\,\right]$$

where $r \equiv 6\,t_2^2 + t_1^2 + 2\,t_1\,t_2$.

The roots of the equation $5\,\Delta^2 + 10\,t_2\,\Delta + r = 0$ are determined by:

$$\Delta \; = \; \frac{1}{10}\left[\, -10\,t_2 \pm \sqrt{100\,t_2^2 - 20\,(6\,t_2^2 + t_1^2 + 2\,t_1\,t_2)}\,\right]$$

$$= \; - \; t_2 \pm \frac{1}{10}\sqrt{-\,(\,20\,t_2^2 + 20\,t_1^2 + 40\,t_1\,t_2\,)} \; = \; - \; t_2 \pm \frac{i\,[\,t_1 + t_2\,]\,\sqrt{20}}{10}\,.$$

Therefore, the equation has no real roots. Furthermore, $5\,\Delta^2 + 10\,t_2\,\Delta + r|_{\Delta = 0} = r > 0$. It is also readily verified that $5\,\Delta^2 + 10\,t_2\,\Delta + r > 0$ when Δ is evaluated at its extreme values specified in Assumption 2. Therefore

$$\frac{\partial W^*}{\partial t_1} \stackrel{s}{=} -\frac{1}{2}\left[\,5\,\Delta^2 + 10\,t_2\,\Delta + r\,\right] < 0 \;\text{ for all feasible values of } \Delta. \quad \blacksquare$$

Observation 6 indicates that even though the profit of both firms may increase as consumer transportation costs to one firm increase, welfare will decline (due to the associated reduction in consumers' surplus).

We now record in this setting with asymmetric transportation costs the counterparts to the formal conclusions in section 4.

Proposition 6. *Suppose* $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2}$. *Then in the setting with asymmetric transportation costs, a firm will win an auction for the input increment if and only if welfare is highest when the firm secures the input.*

Proof. The conclusion follows from (48) and (60) if $\frac{1}{2}\,[\,t_1 - t_2\,] = \widetilde{D}$. Observe that:

$$\frac{1}{2}\,[\,t_1 - t_2\,] \; - \; \widetilde{D} \; = \; \frac{1}{5\left[\,\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\,\right]}\left\{\frac{5}{2}\,[\,t_1 - t_2\,]\left[\,\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\,\right] - \frac{\partial m_2}{\partial k_2}\,[\,7\,t_1 + 2\,t_2\,] + \frac{\partial m_1}{\partial k_1}\,[\,7\,t_2 + 2\,t_1\,]\right\}$$

$$= \frac{1}{5\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]} \left\{ \frac{\partial m_1}{\partial k_1}\left[7\,t_2 + 2\,t_1 + \frac{5}{2}(\,t_1 - t_2\,)\right] \right.$$

$$\left. - \frac{\partial m_2}{\partial k_2}\left[7\,t_1 + 2\,t_2 - \frac{5}{2}(\,t_1 - t_2\,)\right]\right\}$$

$$= \frac{1}{5\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]} \left\{ \frac{\partial m_1}{\partial k_1}\left[\frac{9\,t_2 + 9\,t_1}{2}\right] - \frac{\partial m_2}{\partial k_2}\left[\frac{9\,t_1 + 9\,t_2}{2}\right]\right\}$$

$$= \frac{9\,[\,t_1 + t_2\,]}{10\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]}\left[\frac{\partial m_1}{\partial k_1} - \frac{\partial m_2}{\partial k_2}\right] \equiv G\,. \tag{61}$$

It is apparent that $G = 0$ when $\frac{\partial m_1}{\partial k_1} = \frac{\partial m_2}{\partial k_2}$. ■

Proposition 7. *Suppose* $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$ *and* $\Delta \equiv m_1 - m_2 \in (\,\widetilde{D}\,, \frac{1}{2}\,[\,t_1 - t_2\,]\,)$. *Then in the setting with asymmetric transportation costs, firm 1 will not win an auction for the input increment even though welfare would be highest if it did win the auction.*

Proof. From (48), firm 1 wins the auction if $\Delta > \frac{1}{2}\,[\,t_1 - t_2\,]$. From (59), welfare is highest when firm 1 wins the auction if $\Delta > \widetilde{D}$. Therefore, firm 1 will not win an auction for the input increment even though welfare would be highest if it did win the auction when $\Delta \in (\,\widetilde{D}, \frac{1}{2}\,[\,t_1 - t_2\,])$. Observe from (61) that $\widetilde{D} < \frac{1}{2}\,[\,t_1 - t_2\,]$ if and only if $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$. ■

Proposition 8. *Suppose* $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$ *and* $\Delta \equiv m_1 - m_2 \in (\,\frac{1}{2}\,[\,t_1 - t_2\,]\,, \widetilde{D}\,)$. *Then in the setting with asymmetric transportation costs, firm 1 will win an auction for the input increment even though welfare is highest when it does not win the auction.*

Proof. From (48), firm 1 wins the auction if $\Delta > \frac{1}{2}\,[\,t_1 - t_2\,]$. From (59), welfare is highest when firm 1 wins the auction if $\Delta > \widetilde{D}$. Therefore, firm 1 will the auction even though welfare would be highest if it did not win the auction when $\Delta \in (\,\frac{1}{2}\,[\,t_1 - t_2\,]\,, \widetilde{D}\,)$. Observe from (61) that $\frac{1}{2}\,[\,t_1 - t_2\,] < \widetilde{D}$ if and only if $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$. ■

Proposition 9. *In the setting with asymmetric transportation costs, the set of values of* $\Delta \equiv m_1 - m_2$ *for which an unfettered auction fails to implement the welfare-maximizing allocation of an input increment declines as* t_1 *or* t_2 *declines, ceteris paribus.*

Proof. From Propositions 7 and 8, the set of values of Δ for which an unfettered auction fails to implement the welfare-maximizing allocation of the input increment is: (i) $\Delta \in$

$(\widetilde{D}, \frac{1}{2}[t_1 - t_2])$ when $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$; and (ii) $\Delta \in (\frac{1}{2}[t_1 - t_2], \widetilde{D})$ when $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$. From (61), when $\frac{\partial m_1}{\partial k_1} > \frac{\partial m_2}{\partial k_2}$:

$$\frac{\partial}{\partial t_i}\left(\frac{1}{2}[t_1 - t_2] - \widetilde{D}\right) = \frac{9\left[\frac{\partial m_1}{\partial k_1} - \frac{\partial m_2}{\partial k_2}\right]}{10\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]} > 0 \text{ for } i = 1, 2.$$

(61) also implies that when $\frac{\partial m_2}{\partial k_2} > \frac{\partial m_1}{\partial k_1}$:

$$\frac{\partial}{\partial t_i}\left(\widetilde{D} - \frac{1}{2}[t_1 - t_2]\right) = \frac{9\left[\frac{\partial m_2}{\partial k_2} - \frac{\partial m_1}{\partial k_1}\right]}{10\left[\frac{\partial m_1}{\partial k_1} + \frac{\partial m_2}{\partial k_2}\right]} > 0 \text{ for } i = 1, 2. \quad \blacksquare$$

Proposition 10. *In the setting with asymmetric transportation costs, an unfettered auction will never allocate an input increment in a manner that reduces welfare below the level that prevails in the absence of the increment.*

Proof. Equations (48) and (55) imply that if $\Delta > \frac{1}{2}[t_1 - t_2]$, then firm 1 wins the auction and

$$\frac{\partial W^*}{\partial k_1} = \frac{\partial m_1}{\partial k_1}\left[\frac{7t_2 + 2t_1 + 5\Delta}{9(t_1 + t_2)}\right] > \frac{\partial m_1}{\partial k_1}\left[\frac{7t_2 + 2t_1 + \frac{5}{2}(t_1 - t_2)}{9(t_1 + t_2)}\right] = \frac{1}{2}\frac{\partial m_1}{\partial k_1} > 0.$$

Equations (48) and (56) imply that if $\Delta < \frac{1}{2}[t_1 - t_2]$, then firm 2 wins the auction and

$$\frac{\partial W^*}{\partial k_2} = \frac{\partial m_2}{\partial k_2}\left[\frac{7t_1 + 2t_2 - 5\Delta}{9(t_1 + t_2)}\right] > \frac{\partial m_2}{\partial k_2}\left[\frac{7t_1 + 2t_2 - \frac{5}{2}(t_1 - t_2)}{9(t_1 + t_2)}\right] = \frac{1}{2}\frac{\partial m_2}{\partial k_2} > 0.$$

If $\Delta = \frac{1}{2}[t_1 - t_2]$, then equations (55) and (56) imply that $\frac{\partial W^*}{\partial k_i} = \frac{1}{2}\frac{\partial m_i}{\partial k_i} > 0$ for $i = 1, 2$.
$\blacksquare$

B. The Setting with Value-Enhancing Effort.

In this setting, $v_i(k_i, e_i)$ will denote the value that consumers place on firm i's product when the firm employs k_i units of the input and devotes effort e_i to reducing its unit production cost and enhancing consumer valuation of its product. Furthermore, $c_i(k_i, e_i)$ will denote firm i's corresponding unit cost of production. We assume that $v_i(\cdot)$ is increasing in e_i and $c_i(\cdot)$ is decreasing in e_i. The firms choose their effort supplies (simultaneously and independently) after the conclusion of the input auction, prior to setting prices. Firm $i \in \{1, 2\}$ incurs personal cost $E_i(e_i)$ when it delivers effort e_i. $E_i(\cdot)$ is an increasing, convex function of e_i.

From (8), firm 1's equilibrium profit in this setting is of the form:

$$\pi_1^*(\cdot) = \frac{1}{18t}[3t + \Delta]^2 - E_1(e_1). \tag{62}$$

Differentiating (62) provides:

$$\frac{\partial \pi_1^*(\cdot)}{\partial k_1} = \frac{1}{9\,t}\,[\,3\,t + \Delta\,]\,\frac{d\Delta}{dk_1} - E_1'(e_1)\,\frac{de_1}{dk_1} \quad \text{and}$$

$$\frac{\partial \pi_1^*(\cdot)}{\partial k_2} = \frac{1}{9\,t}\,[\,3\,t + \Delta\,]\,\frac{d\Delta}{dk_2} - E_1'(e_1)\,\frac{de_1}{dk_2}\,. \tag{63}$$

(63) implies that the rate at which firm 1's profit increases as it, rather than firm 2, secures more of the input is:

$$\widehat{B}_1 = \frac{\partial \pi_1^*(\cdot)}{\partial k_1} - \frac{\partial \pi_1^*(\cdot)}{\partial k_2} = \frac{1}{9\,t}\,[\,3\,t + \Delta\,]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right]. \tag{64}$$

Analogous calculations for firm 2 reveal that the rate at which firm 2's profit increases as it, rather than firm 1, secures more of the input is:

$$\widehat{B}_2 = \frac{\partial \pi_2^*(\cdot)}{\partial k_2} - \frac{\partial \pi_2^*(\cdot)}{\partial k_1} = \frac{1}{9\,t}\,[\,3\,t - \Delta\,]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_2'(e_2)\left[\frac{de_2}{dk_2} - \frac{de_2}{dk_1}\right]. \tag{65}$$

(64) and (65) imply that firm 1 will outbid firm 2 for the input increment in an unfettered auction if and only if:

$$\widehat{B}_1 > \widehat{B}_2 \;\Leftrightarrow\; \frac{1}{9\,t}\,[\,3\,t + \Delta\,]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right]$$

$$> \frac{1}{9\,t}\,[\,3\,t - \Delta\,]\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] - E_2'(e_2)\left[\frac{de_2}{dk_2} - \frac{de_2}{dk_1}\right]$$

$$\Leftrightarrow\; \frac{2\,\Delta}{9\,t}\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] > E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right] - E_2'(e_2)\left[\frac{de_2}{dk_2} - \frac{de_2}{dk_1}\right]$$

$$\Leftrightarrow\; \Delta > \frac{9\,t}{2}\left[\frac{E_1'(e_1)\left(\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right) - E_2'(e_2)\left(\frac{de_2}{dk_2} - \frac{de_2}{dk_1}\right)}{\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}}\right]. \tag{66}$$

From (12), aggregate welfare in this setting is:

$$\widehat{W}^* = \frac{m_1}{6\,t}\,[\,3\,t + \Delta\,] + \frac{m_2}{6\,t}\,[\,3\,t - \Delta\,] - \frac{t}{4} - \frac{\Delta^2}{36\,t} - E_1(e_1) - E_2(e_2)$$

$$\Rightarrow\; \frac{\partial \widehat{W}^*}{\partial k_1} = \frac{m_1}{6\,t}\,\frac{d\Delta}{dk_1} + \left[\frac{3\,t + \Delta}{6\,t}\right]\frac{dm_1}{dk_1} - \frac{m_2}{6\,t}\,\frac{d\Delta}{dk_1} + \left[\frac{3\,t - \Delta}{6\,t}\right]\frac{dm_2}{dk_1} - \frac{\Delta}{18\,t}\,\frac{d\Delta}{dk_1}$$

$$- E_1'(e_1)\,\frac{de_1}{dk_1} - E_2'(e_2)\,\frac{de_2}{dk_2}$$

$$= \frac{\Delta}{6\,t}\frac{d\Delta}{dk_1} - \frac{\Delta}{18\,t}\frac{d\Delta}{dk_1} + \frac{1}{2}\left[\frac{dm_1}{dk_1} + \frac{dm_2}{dk_1}\right] + \frac{\Delta}{6\,t}\left[\frac{dm_1}{dk_1} - \frac{dm_2}{dk_1}\right]$$

$$- E_1'(e_1)\frac{de_1}{dk_1} - E_2'(e_2)\frac{de_2}{dk_2}$$

$$= \frac{5\,\Delta}{18\,t}\frac{d\Delta}{dk_1} + \frac{1}{2}\left[\frac{dm_1}{dk_1} + \frac{dm_2}{dk_1}\right] - E_1'(e_1)\frac{de_1}{dk_1} - E_2'(e_2)\frac{de_2}{dk_2}\,. \quad (67)$$

Similarly:

$$\frac{\partial \widehat{W}^*}{\partial k_2} = \frac{m_1}{6\,t}\frac{d\Delta}{dk_2} + \left[\frac{3\,t+\Delta}{6\,t}\right]\frac{dm_1}{dk_2} - \frac{m_2}{6\,t}\frac{d\Delta}{dk_2} + \left[\frac{3\,t-\Delta}{6\,t}\right]\frac{dm_2}{dk_2} - \frac{\Delta}{18\,t}\frac{d\Delta}{dk_2}$$

$$- E_1'(e_1)\frac{de_1}{dk_2} - E_2'(e_2)\frac{de_2}{dk_2}$$

$$= \frac{\Delta}{6\,t}\frac{d\Delta}{dk_2} - \frac{\Delta}{18\,t}\frac{d\Delta}{dk_2} + \frac{1}{2}\left[\frac{dm_1}{dk_2} + \frac{dm_2}{dk_2}\right] + \frac{\Delta}{6\,t}\left[\frac{dm_1}{dk_2} - \frac{dm_2}{dk_2}\right]$$

$$- E_1'(e_1)\frac{de_1}{dk_2} - E_2'(e_2)\frac{de_2}{dk_2}$$

$$= \frac{5\,\Delta}{18\,t}\frac{d\Delta}{dk_2} + \frac{1}{2}\left[\frac{dm_1}{dk_2} + \frac{dm_2}{dk_2}\right] - E_1'(e_1)\frac{de_1}{dk_2} - E_2'(e_2)\frac{de_2}{dk_2}\,. \quad (68)$$

(67) and (68) imply that the rate at which welfare increases as firm 1, rather than firm 2, acquires more of the input is:

$$\widehat{G}_1 = \frac{\partial W^*(\cdot)}{\partial k_1} - \frac{\partial W^*(\cdot)}{\partial k_2} = \frac{5\,\Delta}{18\,t}\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] + \frac{1}{2}\left[\frac{d\,(m_1+m_2)}{dk_1} - \frac{d\,(m_1+m_2)}{dk_2}\right]$$

$$- E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right] - E_2'(e_2)\left[\frac{de_2}{dk_1} - \frac{de_2}{dk_2}\right]. \quad (69)$$

Similarly, the rate at which welfare increases as firm 2, rather than firm 1, acquires more of the input is:

$$\widehat{G}_2 = \frac{\partial W^*(\cdot)}{\partial k_2} - \frac{\partial W^*(\cdot)}{\partial k_1} = -\,\widehat{G}_1\,. \quad (70)$$

(69) and (70) imply that welfare increases more rapidly when firm 1, rather than firm 2, acquires more of the input if and only if:

$$\widehat{G}_1 > \widehat{G}_2 \;\Leftrightarrow\; \widehat{G}_1 > -\,\widehat{G}_1 \;\Leftrightarrow\; \widehat{G}_1 > 0$$

$$\Leftrightarrow\; \frac{5\,\Delta}{18\,t}\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] > \frac{1}{2}\left[\frac{d\,(m_1+m_2)}{dk_2} - \frac{d\,(m_1+m_2)}{dk_1}\right]$$

$$+ \; E_1'(e_1)\left[\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right] + E_2'(e_2)\left[\frac{de_2}{dk_1} - \frac{de_2}{dk_2}\right]$$

$$\Leftrightarrow \quad \Delta\left[\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}\right] \; > \; \frac{9\,t}{5}\left[\frac{d\,(m_1+m_2)}{dk_2} - \frac{d\,(m_1+m_2)}{dk_1}\right] + \frac{18\,t}{5}\left[E_1'(e_1)\left(\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right) + E_2'(e_2)\left(\frac{de_2}{dk_1} - \frac{de_2}{dk_2}\right)\right]$$

$$\Leftrightarrow \quad \Delta \; > \; \frac{9\,t}{5}\left[\frac{\frac{d(m_1+m_2)}{dk_2} - \frac{d(m_1+m_2)}{dk_1}}{\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}}\right] + \frac{18\,t}{5}\left[\frac{E_1'(e_1)\left(\frac{de_1}{dk_1} - \frac{de_1}{dk_2}\right) + E_2'(e_2)\left(\frac{de_2}{dk_1} - \frac{de_2}{dk_2}\right)}{\frac{d\Delta}{dk_1} - \frac{d\Delta}{dk_2}}\right]. \tag{71}$$

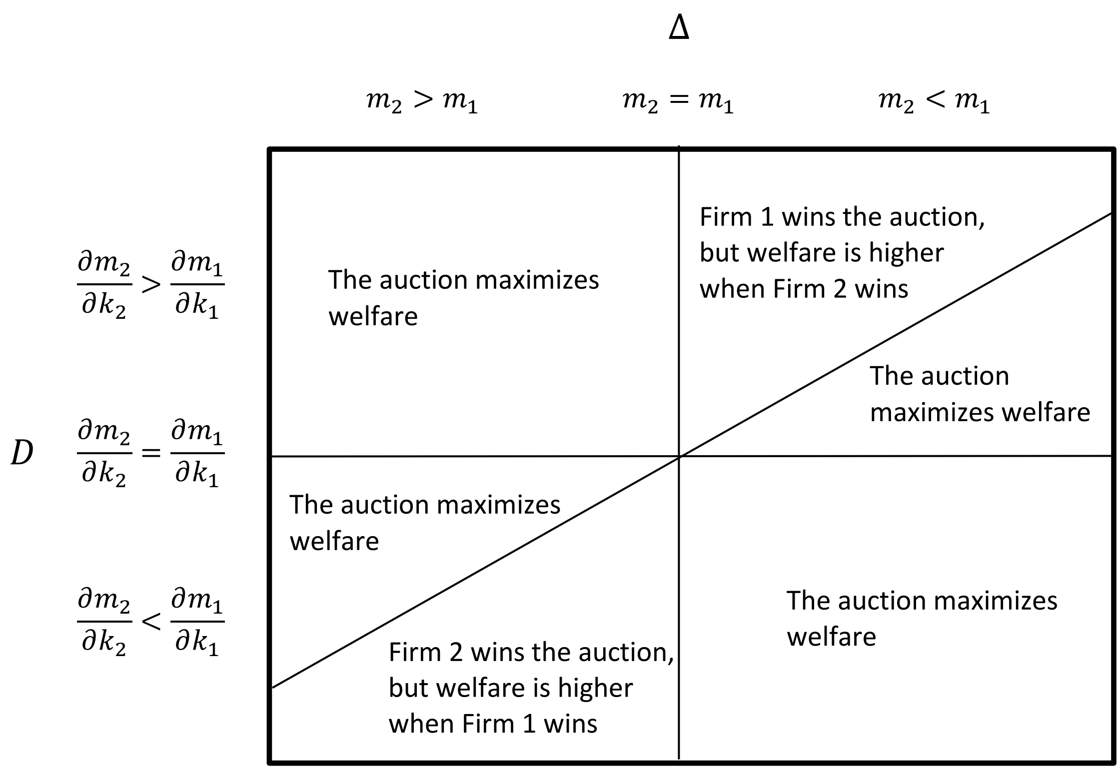

Figure 1. Outcomes under Unfettered Input Auctions.

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